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Classification of bicovariant differential calculi on the Jordanian quantum groups $GL_{h,g}(2)$ and $SL_h(2)$ and quantum Lie algebras

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Abstract. Assuming that the bicovariant bimodules are generated as left modules by the differentials of the quantum group generators we classify all four-dimensional first-order bicovariant calculi on the Jordanian quantum group $GL_{h,g}(2)$ and all three-dimensional first-order bicovariant calculi on the Jordanian quantum group $SL_h(2)$. It is found that there are three one-parameter families of four-dimensional bicovariant first-order calculi on $GL_{h,g}(2)$ and that there is a single, unique, three-dimensional bicovariant calculus on $SL_h(2)$. This three-dimensional calculus may be obtained through a classical-like reduction from any one of the three families of four-dimensional calculi on $GL_{h,g}(2)$. Details of the higher order calculi and also the quantum Lie algebras are presented for all calculi. The quantum Lie algebra obtained from the bicovariant calculus on $SL_h(2)$ is shown to be isomorphic to the quantum Lie algebra we obtain as an ad-submodule within the Jordanian universal enveloping algebra $U_h(\mathfrak{sl}_2(\mathbb{C}))$ and also through a consideration of the decomposition of the tensor product of two copies of the deformed adjoint module. We also obtain the quantum Killing form for this quantum Lie algebra.

1. Introduction

The programme of noncommutative geometry pioneered by Connes [1, 2] is based on fundamental results in the field of abstract analysis discovered in the first half of this century by Gelfand, Kolmogoroff, Naimark, Stone and others (a useful historical overview can be found in Segal's review [3] of Connes' book [1]). In particular, Gelfand and Kolmogoroff showed that for a locally compact space, the algebra of continuous functions on the space is essentially *equivalent* to the space itself. The algebra of continuous functions is of course commutative, and in fact a C^*-algebra. We can then reasonably consider the study of *noncommutative* C^*-algebras as some form of noncommutative geometry. Thus the essential idea is to express the formalism of classical geometry as far as possible in the language of commutative algebra, and then use this as the paradigm for generalizing to the noncommutative setting.

An implementation of this programme has been developed by Dubois-Violette and co-workers (see the book by Madore [5], and the references therein). They generalize an elegant algebraic approach to the differential geometry of a smooth manifold introduced by

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Koszul [4] to the case where the commutative algebra of smooth functions on the manifold is replaced by the noncommutative algebra of matrices over some field.

In another direction, quantum groups provide natural candidates for noncommutative generalizations of the algebras of smooth functions on classical compact Lie groups. Classically the algebra of *representative functions* is a dense subalgebra of the algebra of all smooth complex-valued functions on the group and carries the structure of a Hopf $\star$-algebra. Dropping the $\star$-structure we obtain the coordinate rings of the corresponding complex Lie groups, $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$. These coordinate rings are generated already by the matrix elements of the defining representations of these groups (for $GL_n(\mathbb{C})$ we should adjoin $(\det(\boldsymbol{t}))^{-1}$ where $\boldsymbol{t}$ is the matrix of matrix element functions). Corresponding to these classical groups are the well known FRT Hopf algebras $A(R)$, introduced by Faddeev, Reshetikhin and Takhtajan (FRT) in [50]. They are 'quantizations' of the classical coordinate rings. It is these structures upon which we should try to develop some sort of noncommutative Lie group geometry. The pioneering works here are those of Woronowicz [6, 7]. In particular, in [7] Woronowicz set out a formalism in terms of *bicovariant bimodules* which has been studied intensively by very many authors since. Let us note that the classical differential calculus on Lie groups is bicovariant.

There is a well known 'problem' with the bicovariant calculi associated with the standard FRT quantum groups other than $GL_q(n)$: their dimensions do not agree with the corresponding classical calculi. In the particular case of $SL_q(n)$ the bicovariant calculus of Woronowicz is n^2-dimensional while the classical calculus is of dimension $n^2 - 1$ (the dimension being the dimension of the vector space of left-invariant one-forms). This problem has stimulated some authors to consider alternative approaches to the development of differential geometry on quantum groups. For example, in [35] Schmüdgen and Schüler consider left-covariant bimodules (developing Woronowicz's original approach [6]) on $SL_q(n)$. They obtain first-order calculi with the classical dimension. However, for $N \geqslant 4$ the higher-order calculi do not have the correct dimension. Another interesting approach was initiated by Faddeev and Pyatov [36]. They considered, for the particular case of $SL_q(n)$, the consequences of relaxing the condition of the classical Leibniz rule which is present in the bicovariant Woronowicz approach. They obtained bicovariant calculi of the correct classical dimension at all orders. However, subsequent work by Arutyunov *et al* [11] suggests that a similar approach cannot be employed for the other simple quantum groups $SO_q(n)$ and $Sp_q(n)$.

In this paper we consider the original Woronowicz bicovariant calculus, but we examine such calculi on the *nonstandard* quantum groups $GL_{g,h}(2)$ and $SL_h(2)$—the so-called Jordanian quantum groups. In [33, 34], Karimpour initiated the study of bicovariant calculi associated with $SL_h(2)$. Here, working under the assumption that the bicovariant bimodules are generated as left modules by the differentials of the quantum group generators, we perform a complete classification of all first-order bicovariant calculi on the quantum groups $GL_{h,g}(2)$ and $SL_h(2)$. Furthermore we consider the higher-order calculi and the corresponding quantum Lie algebras. Let us summarize our main classification results.

- There are three one-parameter families of four-dimensional first-order bicovariant differential calculi on $GL_{h,g}(2)$ whose bimodules of forms are generated as left $GL_{h,g}(2)$-modules by the differentials of the quantum group generators.
- For one value of the parameter, the calculi in the three families are the same. This parameter value coincides with the value required for a 'classical-like' reduction to a three-dimensional first-order bicovariant calculus on $SL_h(2)$ which is shown to be unique.
- For all the calculi the relations in the exterior algebra are obtained and are shown to lead to exterior calculi whose dimension is classical at all orders.

• For all the calculi the relations in the enveloping algebra of the quantum Lie algebra are obtained and are shown to lead to PBW-type bases.

Classically, the Lie algebra of a Lie group is obtained as the vector space of tangent vectors at the identity equipped with a Lie bracket defined in terms of the left-invariant vector fields on the group manifold. The formalism of Woronowicz's bicovariant calculus has a natural construction for a 'quantum Lie algebra' generalizing the classical construction to the abstract Hopf algebra setting. However, for all standard quantum groups the quantum Lie algebras so obtained have the 'wrong' dimension. This has prompted authors such as Sudbery and Delius to look for alternative constructions for quantum Lie algebras [43–45]. There are two different approaches described in their work. Recall that classically the Lie algebra $\mathfrak{g}$ is an ad-submodule of the classical adjoint $U(\mathfrak{g})$-module, $U(\mathfrak{g})$, and its Lie bracket is the restriction of this classical adjoint action to $\mathfrak{g}$. This motivates the first approach [43, 44], *approach 1*, in which we look for an ad-submodule *within* the quantized universal enveloping algebra $U_q(\mathfrak{g})$ which has the correct dimension and upon which the restriction of the adjoint action of $U_q(\mathfrak{g})$ closes. Lyubashenko and Sudbery employed a result of Joseph and Letzter and obtained a quantum Lie algebra in the cases $U_q(\mathfrak{sl}_n(\mathbb{C}))$ such that the coproduct of $U_q(\mathfrak{sl}_n(\mathbb{C}))$ applied to the basis of their quantum Lie algebra is of a particularly neat form. The other approach [45], *approach 2*, (see also the paper by Bremner [47] and the earlier paper of Donin and Gurevich [48] where the idea appeared originally) constructs a quantum Lie algebra independent of any embedding into $U_q(\mathfrak{g})$. The idea here is to recall that classically the Lie bracket is an intertwiner, $[\,,\,] : \mathrm{ad} \otimes \mathrm{ad} \to \mathrm{ad}$, where ad is the usual adjoint representation of $U(\mathfrak{g})$. The quantum Lie bracket is then obtained as the intertwiner of the corresponding $U_q(\mathfrak{g})$-modules. Furthermore, classically the Killing form is an intertwiner, $\mathfrak{B} : \mathrm{ad}\otimes\mathrm{ad} \to \mathbb{C}$, and considering the intertwiner between the tensor product of the two (quantum) adjoint modules and the trivial representation, an analogue of the Killing form for the quantum Lie algebras is obtained in a rather straightforward manner. The two approaches lead to isomorphic quantum Lie algebras. However, each has advantages over the other: *approach 1* allows us to see explicitly the relationship between the quantum Lie algebra and the quantized universal enveloping algebra and, in principle, allows us to determine the coproduct of $U_q(\mathfrak{g})$ on the quantum Lie algebra; *approach 2* gives a reasonably simple prescription for constructing quantum Lie algebras based on computation of inverse Clebsch–Gordan coefficients [46].

In the last part of this paper we pursue this line of enquiry starting with the Jordanian quantized enveloping algebra $U_h(\mathfrak{sl}_2(\mathbb{C}))$. The following results are obtained.

• A Jordanian quantum Lie algebra is obtained according to *approach 1* and the expressions for the coproduct of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ on the basis elements of the quantum Lie algebra are obtained. A definition for an invariant Killing form is also recalled and its evaluations on the quantum Lie algebra elements is presented.

• The details of the motivation for *approach 2* are recalled and this approach is used to check the commutators and Killing form obtained already through *approach 1*.

• We observe that the quantum Lie algebras obtained equivalently by *approach 1 and approach 2* are isomorphic to the quantum Lie algebra obtained through the unique bicovariant calculus on the Jordanian quantum group $SL_h(2)$.

The paper is self-contained and organized as follows. In section 2 we recall pertinent definitions and results concerning the Jordanian quantum groups $GL_{h,g}(2)$ and $SL_h(2)$. In section 3 Woronowicz's theory is reviewed in a style intended to enlighten the details of the classification procedure, due to Müller-Hoissen [22], which we present in section 4. The classification results appear in sections 5 and 6. The Jordanian quantized universal enveloping algebra is recalled in section 7 with the corresponding quantum Lie algebras obtained in sections 8 and 9 through *approach 1* and *approach 2* respectively. In the final section 10 we complete the picture by observing that these quantum Lie algebras are isomorphic to the one obtained from the bicovariant calculus on the Jordanian quantum group $SL_h(2)$.

2. The Jordanian quantum groups

The two-parameter Jordanian quantum group $GL_{h,g}(2)$ is the co-quasitriangular Hopf algebra derived from the following R-matrix,

$$R = \begin{pmatrix} 1 & -h & h & gh \\ 0 & 1 & 0 & -g \\ 0 & 0 & 1 & g \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Being triangular, i.e. $R_{21}R = I$, this R-matrix is trivially Hecke, with

$$(\hat{R} - 1)(\hat{R} + 1) = 0 \tag{2}$$

where $\hat{R} = PR$ and $P_{ij,kl} = \delta_{il}\delta_{jk}$.

The quantum group associated with the R-matrix $R_{21} = R(-h, -g)$, with $g = h = 1$, was first investigated by Demidov *et al* [8], while R_{21} with $g = h$ is the one-parameter nonstandard R-matrix whose quantum group was considered by Zakrzewski [9]. Lazarev and Movshev [15] considered the quantum group associated with R with $g = h$ and also the corresponding quantized universal enveloping algebra, $U_h(\mathfrak{sl}_2(\mathbb{C}))$. The quantized enveloping algebra was also investigated by Ohn [19] and will be discussed in more detail in section 7. In fact, the R-matrix, (1), can be extracted from an early work of Gurevich [10]; though the associated quantum group structure was not investigated there.

In the usual way, defining an algebra valued matrix T as

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3}$$

the relations of a matrix element bialgebra $A(R)$ are obtained from the well known FRT [50] matrix relation,

$$RT_1T_2 = T_2T_1R \tag{4}$$

as

$$\begin{aligned} &ca = ac - gc^2 \qquad cd = dc - hc^2 \\ &db = bd + g(ad - bc + hac - d^2) \\ &ab = ba + h(ad - bc + hac - a^2) \\ &cb = bc - hac - gdc + ghc^2 \qquad da = ad + hac - gdc. \end{aligned} \tag{5}$$

The coalgebra structure is provided by a coproduct defined on the generators as,

$$\begin{aligned} &\Delta(a) = a \otimes a + b \otimes c \qquad \Delta(b) = a \otimes b + b \otimes d \\ &\Delta(c) = c \otimes a + d \otimes c \qquad \Delta(d) = c \otimes b + d \otimes d \end{aligned} \tag{6}$$

with the counit given by,

$$\begin{aligned} &\epsilon(a) = 1 \qquad \epsilon(b) = 0 \\ &\epsilon(c) = 0 \qquad \epsilon(d) = 1. \end{aligned} \tag{7}$$

$\hat{R}$ has a spectral decomposition,

$$\hat{R} = P^{+} + P^{-} \tag{8}$$

where $P^{+} = \frac{1}{2}(\hat{R} + I)$ is a rank 3 projector and $P^{-} = \frac{1}{2}(\hat{R} - I)$ is a rank 1 projector. In the notation of Majid [12], these projectors provide the associated quantum coplane, $\mathbb{A}_{-1}^{2|0}$, and plane, $\mathbb{A}_{1}^{2|0}$, respectively through the relations $P^{\pm}\boldsymbol{x}_1\boldsymbol{x}_2 = 0$ where $\boldsymbol{x}$ is the 2×1 column vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. These are associative algebras generated by the elements x_1 and x_2 subject to the relations,

$$x_2^2 = 0 \qquad x_1^2 = -hx_2x_1 \qquad x_1x_2 = -x_2x_1 \tag{9}$$

in the case of the coplane $\mathbb{A}_{-1}^{2|0}$, and,

$$x_1x_2 = x_2x_1 + gx_2^2 \tag{10}$$

in the case of the plane $\mathbb{A}_{1}^{2|0}$. A result of Mukhin [14, theorems 1–3] (see also [49, theorem 3.5]) then tells us that $A(R)$ is the *universal coacting bialgebra* on this pair of algebras in the sense of Manin [13]. In the language of Sudbery [16], $\mathbb{A}_{-1}^{2|0}$ and $\mathbb{A}_{1}^{2|0}$ are then *complementary coordinate algebras* determining $A(R)$, and with an easy application of the diamond lemma [17] telling us that they are moreover *superpolynomial algebras* generated by odd and even generators respectively having ordering algorithms with respect to the ordering of the generators, $x_2 \prec x_1$, we deduce immediately from a result of Sudbery [16, theorem 3] that $A(R)$ has as a basis the ordered monomials $\{b^{\alpha}a^{\beta}d^{\gamma}c^{\delta} : \alpha, \beta, \gamma, \delta \in \mathbb{Z}_{\geqslant 0}\}$. This fact is used extensively in the computations which lead to our main results.

It follows from $\mathbb{A}_{-1}^{2|0}$, that R is *Frobenius* [40] and so in the usual way we can obtain a group-like element in the bialgebra $A(R)$, $\mathcal{D}$, called the quantum determinant, and given by,

$$\mathcal{D} = ad - bc + hac. \tag{11}$$

The commutation relations between $\mathcal{D}$ and the generators of $A(R)$ are,

$$\begin{aligned} &\mathcal{D}a = a\mathcal{D} + (h-g)c\mathcal{D} \qquad \mathcal{D}d = d\mathcal{D} - (h-g)c\mathcal{D} \\ &\mathcal{D}c = c\mathcal{D} \qquad \mathcal{D}b = b\mathcal{D} + (h-g)(d\mathcal{D} - a\mathcal{D} - (h-g)c\mathcal{D}) \end{aligned} \tag{12}$$

so we can localize with respect to the Ore set [55] $S = \{\mathcal{D}^{\alpha} : \alpha \in \mathbb{Z}_{\geqslant 1}\}$, and define $GL_{h,g}(2) = A(R)[\mathcal{D}^{-1}]$, having extra commutation relations,

$$\begin{aligned} &a\mathcal{D}^{-1} = \mathcal{D}^{-1}a + (h-g)\mathcal{D}^{-1}c \qquad d\mathcal{D}^{-1} = \mathcal{D}^{-1}d - (h-g)\mathcal{D}^{-1}c \\ &c\mathcal{D}^{-1} = \mathcal{D}^{-1}c \qquad b\mathcal{D}^{-1} = \mathcal{D}^{-1}b + (h-g)(\mathcal{D}^{-1}d - \mathcal{D}^{-1}a - (h-g)\mathcal{D}^{-1}c) \end{aligned} \tag{13}$$

with

$$\Delta(\mathcal{D}^{-1}) = \mathcal{D}^{-1} \otimes \mathcal{D}^{-1} \qquad \epsilon(\mathcal{D}^{-1}) = 1. \tag{14}$$

$GL_{h,g}(2)$ is a Hopf algebra with the antipode given by

$$\begin{aligned} &S(a) = \mathcal{D}^{-1}(d + gc) \qquad S(b) = \mathcal{D}^{-1}(gd - ga - b + g^2c) \\ &S(c) = -\mathcal{D}^{-1}c \qquad S(d) = \mathcal{D}^{-1}(a - gc) \qquad S(\mathcal{D}^{-1}) = \mathcal{D}. \end{aligned} \tag{15}$$

The Hopf algebra $GL_{h,g}(2)$ is clearly still polynomial with basis $\{\mathcal{D}^{-\alpha}b^{\beta}a^{\gamma}d^{\delta}c^{\zeta}: \alpha, \beta, \gamma, \delta, \zeta \in \mathbb{Z}_{\geqslant 0}\}$.

With $g = h$, $\mathcal{D}$ is central and we can consistently set $\mathcal{D} = 1$ and pass to the quantum group $SL_h(2)$. The relations for $SL_h(2)$ are just (5), but with the combination ad replaced wherever it appears by $bc - hac + 1$ and also the further relation $ad = bc - hac + 1$. With $g = h = 0$ we recover the classical group coordinate rings.

3. Review of Woronowicz's bicovariant differential calculus

We begin with the basic definitions.

Definition 3.1. A *first-order differential calculus* over an algebra A is a pair (Γ, d) such that:

(1) Γ is an A-bimodule, i.e.

$$(a\omega)b = a(\omega b) \tag{16}$$

for all $a, b \in A$, $\omega \in \Gamma$, where the left and right actions which make Γ, respectively, a left A-module and a right A-module are written multiplicatively;

(2) d is a linear map, $\mathrm{d} : A \to \Gamma$;

(3) for any $a, b \in A$, the Leibniz rule is satisfied, i.e.

$$\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b) \tag{17}$$

(4) the bimodule Γ, or 'space of one-forms', is spanned by elements of the form $a\mathrm{d}b$, $a, b \in A$.

Remark 3.2. Given two first-order differential calculi over an algebra A, (Γ, d) and (Γ', d'), we say that they are *isomorphic* if there is a bimodule isomorphism $\phi : \Gamma \to \Gamma'$ such that $\phi \circ \mathrm{d} = \mathrm{d}'$.

Remark 3.3. We usually write $\mathrm{d}a$ for $\mathrm{d}(a)$.

Definition 3.4. A *bicovariant bimodule* over a Hopf algebra A is a triple $(\Gamma, \Delta_A^L, \Delta_A^R)$ such that:

(1) Γ is an A-bimodule;

(2) Γ is an A-bicomodule with left and right coactions Δ_A^L and Δ_A^R respectively, i.e.

$$(\mathrm{id} \otimes \Delta_A^L) \circ \Delta_A^L = (\Delta \otimes \mathrm{id}) \circ \Delta_A^L \qquad (\epsilon \otimes \mathrm{id}) \circ \Delta_A^L = \mathrm{id} \tag{18}$$

making Γ a left A-comodule,

$$(\Delta_A^R \otimes \mathrm{id}) \circ \Delta_A^R = (\mathrm{id} \otimes \Delta) \circ \Delta_A^R \qquad (\mathrm{id} \otimes \epsilon) \circ \Delta_A^R = \mathrm{id} \tag{19}$$

making Γ a right A-comodule, and

$$(\mathrm{id} \otimes \Delta_A^R) \circ \Delta_A^L = (\Delta_A^L \otimes \mathrm{id}) \circ \Delta_A^R \tag{20}$$

which is the A-bicomodule property;

(3) the coactions, Δ_A^L and Δ_A^R are bimodule maps, i.e.

$$\Delta_A^L(a\omega b) = \Delta(a)\Delta_A^L(\omega)\Delta(b) \tag{21}$$

$$\Delta_A^R(a\omega b) = \Delta(a)\Delta_A^R(\omega)\Delta(b). \tag{22}$$

Remark 3.5. The Sweedler notation for coproducts in the Hopf algebra A is taken to be $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for all $a \in A$ and is extended to the coactions as $\Delta_A^L(\omega) = \omega_{(A)} \otimes \omega_{(\Gamma)}$ and $\Delta_A^R(\omega) = \omega_{(\Gamma)} \otimes \omega_{(A)}$. In this notation the conditions (18)–(20) become,

$$\omega_{(A)} \otimes (\omega_{(\Gamma)})_{(A)} \otimes (\omega_{(\Gamma)})_{(\Gamma)} = (\omega_{(A)})_{(1)} \otimes (\omega_{(A)})_{(2)} \otimes \omega_{(\Gamma)} \qquad \epsilon(\omega_{(A)})\omega_{(\Gamma)} = \omega \tag{23}$$

$$(\omega_{(\Gamma)})_{(\Gamma)} \otimes (\omega_{(\Gamma)})_{(A)} \otimes \omega_{(A)} = \omega_{(\Gamma)} \otimes (\omega_{(A)})_{(1)} \otimes (\omega_{(A)})_{(2)} \qquad \epsilon(\omega_{(A)})\omega_{(\Gamma)} = \omega \tag{24}$$

$$\omega_{(A)} \otimes (\omega_{(\Gamma)})_{(\Gamma)} \otimes (\omega_{(\Gamma)})_{(A)} = (\omega_{(\Gamma)})_{(A)} \otimes (\omega_{(\Gamma)})_{(\Gamma)} \otimes \omega_{(A)} \tag{25}$$

for all $\omega \in \Gamma$.

Definition 3.6. A *first-order bicovariant differential calculus* over a Hopf algebra A is a quadruple $(\Gamma, \mathrm{d}, \Delta_A^L, \Delta_A^R)$ such that:

(1) (Γ, d) is a first-order differential calculus over A;

(2) $(\Gamma, \Delta_A^L, \Delta_A^R)$ is a bicovariant bimodule over A;

(3) d is both a left and a right comodule map, i.e.

$$(\mathrm{id} \otimes \mathrm{d}) \circ \Delta(a) = \Delta_A^L(\mathrm{d}a) \tag{26}$$

$$(\mathrm{d} \otimes \mathrm{id}) \circ \Delta(a) = \Delta_A^R(\mathrm{d}a) \tag{27}$$

for all $a \in A$.

Remark 3.7. Given a first-order calculus over a Hopf algebra, (Γ, d), (26) and (27) uniquely determine left and right coactions, and hence a bicovariant bimodule structure. However, the *existence* of these coactions and the corresponding bicovariant bimodule is of course *not* guaranteed.

Example 3.8. Given any associative algebra A, we may form a first-order differential calculus over A, (A^2, D), where

$$A^2 = \left\{ \sum_k a_k \otimes b_k \in A \otimes A : \sum_k a_k b_k = 0 \right\} \tag{28}$$

and $\mathrm{D} : A \to A^2$ is given by

$$\mathrm{D}a = 1 \otimes a - a \otimes 1. \tag{29}$$

A^2 has a bimodule structure given, for all $a_k, b_k, c \in A$ by

$$c\left(\sum_k a_k \otimes b_k \right) = \sum_k c a_k \otimes b_k \qquad \left(\sum_k a_k \otimes b_k \right) c = \sum_k a_k \otimes b_k c. \tag{30}$$

The importance of this differential calculus lies in the fact that *any* first-order differential calculus over A, (Γ, d), is isomorphic to one of the form $(A^2/\mathcal{N}, \pi \circ \mathrm{D})$ where $\mathcal{N}$ is the kernel of the surjective map $\pi : A^2 \to \Gamma$ defined by $\pi(\sum_k a_k \otimes b_k) = \sum a_k \mathrm{d} b_k$. For this reason, (A^2, D) is said to be *universal*. Moreover, it is not difficult to check that when A is a Hopf algebra, (A^2, D) is a first-order bicovariant differential calculus.

Definition 3.9. An element ω of a bicovariant bimodule $(\Gamma, \Delta_A^L, \Delta_A^R)$ is said to be *left-invariant* if

$$\Delta_A^L(\omega) = 1 \otimes \omega \tag{31}$$

right-invariant if

$$\Delta_A^R(\omega) = \omega \otimes 1 \tag{32}$$

and *bi-invariant* if it is both left- and right-invariant.

Remark 3.10. Denoting the vector space of all left-invariant elements of Γ by Γ_{inv}, there is a projection $P : \Gamma \to \Gamma_{\mathrm{inv}}$ defined on any element $\omega \in \Gamma$ as,

$$P(\omega) = S(\omega_{(A)})\omega_{(\Gamma)}. \tag{33}$$

If $(\Gamma, \mathrm{d}, \Delta_A^L, \Delta_A^R)$ is a first-order bicovariant differential calculus over A then it is not difficult to see the equivalence of the statements that the differentials generate Γ as a left A-module and that the elements $P(\mathrm{d}a) = S(a_{(1)})\mathrm{d}a_{(2)}$, for all $a \in A$, span the vector space of left-invariant one-forms. Further, for any $a \in A$, $\mathrm{d}a$ may be expressed in terms of left-invariant forms as

$$\mathrm{d}a = a_{(1)} P(\mathrm{d}a_{(2)}). \tag{34}$$

An alternative construction of the universal differential calculus occurs in the situation of particular interest to us. It is described in the following example.

Example 3.11. We start with a Hopf algebra A defined in terms of a finite number of generators, $(T_{ij})_{i,j=1\ldots n}$ say, together with certain relations which are consistent with a PBW-type basis and the usual (matrix-element bialgebra) coproduct and counit. We may then introduce a bimodule Γ_0 as the free A-bimodule on the symbols $\mathrm{d}_0 T_{ij}$ with the linear map $\mathrm{d}_0 : A \to \Gamma$ defined on any element of A by way of the Leibniz rule. (Γ_0, d_0) is then also a universal first-order differential calculus, and being therefore isomorphic to (A^2, D) is certainly bicovariant. Indeed, (26) and (27) specify the coactions, and the space of left-invariant forms, $\Gamma_{0\mathrm{inv}}$, is spanned by all elements of the form $S(a_{(1)})\theta_{ik}a_{(2)}$ where $\theta_{ik} = \sum_j S(T_{ij})\mathrm{d}_0 T_{jk}$ and $a \in A$. If we denote by $(\vartheta_i)_{i\in I}$ a basis for Γ_0, where I is some countably infinite index set, then from (27), there exist v_{ij}s such that the right coaction on the ϑ_is takes the form,

$$\Delta_A^R(\vartheta_i) = \sum_{j\in I} \vartheta_j \otimes v_{ji}. \tag{35}$$

In particular, the n^2 left-invariant elements θ_{ik} span a sub-bicomodule, with

$$\Delta_A^R(\theta_{ik}) = \sum_{s,t=1\ldots n} \theta_{st} \otimes S(T_{is})T_{tk}. \tag{36}$$

At this point in example 3.11 the only relations between algebra and bimodule elements are those coming from the Leibniz rule. Obtaining further relations involves finding a suitable ‘relation space’, $\mathcal{N}$, which can be factored out from Γ_0 while maintaining the bicovariance. In the context of the universal calculus, (A^2, D), theorems 1.5 and 1.8 of [7] tell us that such $\mathcal{N}$ must be of the form $\tau^{-1}(A \otimes \mathcal{R})$ where $\tau^{-1}(a \otimes b) = aS(b_{(1)}) \otimes b_{(2)}$ and $\mathcal{R}$ is a right ideal of A, contained in $\ker\epsilon$, which is stable under the right-adjoint coaction†. Conversely, given a first-order bicovariant differential calculus, $(\Gamma, \mathrm{d}, \Delta_A^L, \Delta_A^R)$, $\mathcal{R}$ may be recovered as the set of all $a \in \ker\epsilon$ such that $P(\mathrm{d}a) = 0$.

It is desirable to classify all bicovariant calculi on a given quantum group which have particular properties. This problem of classification can be regarded as the problem of classifying the ad-invariant ideals $\mathcal{R}$ [37–39]. However, in this paper, following Müller-Hoissen [22], we consider a more ‘hands on’ approach. We look for calculi whose bimodule of one-forms is generated as a *left* A-module by the differentials of the generators (this assumption is also made in [37, 38]). In effect we are passing directly to a class of, as yet unspecified, quotients of Γ_0 which we then wish to constrain by the requirement that the bicovariance is not destroyed. For this approach we need the characterization of bicovariant bimodules which is provided by the following theorem of Woronowicz.

Theorem 3.12. Let $(\Gamma, \Delta_A^L, \Delta_A^R)$ be a bicovariant bimodule over A and let $(\theta_i)_{i\in I}$ be a basis of Γ_{inv}, where I is some countable index set. Then we have:

(1) any element $\omega \in \Gamma$ has a unique expression as

$$\omega = \sum_{i\in I} a_i\theta_i \tag{37}$$

where $a_i \in A$;

(2) there exist linear functionals $f_{ij} \in A^*$, $i, j \in I$, such that

$$\theta_i a = \sum_{j\in I} (f_{ij} \star a)\theta_j \tag{38}$$

† We recall that the right-adjoint coaction is defined on any $a \in A$ as $Ad_R^*(a) = a_{(2)} \otimes S(a_{(1)})a_{(3)}$.

where $\alpha \star a = \alpha(a_{(2)})a_{(1)}$ for all $\alpha \in A^*$, $a \in A$;

(3) the functionals f_{ij} are uniquely determined by (38) and satisfy

$$f_{ik}(ab) = \sum_{j\in I} f_{ij}(a) f_{jk}(b) \qquad f_{ik}(1) = \delta_{ik} \tag{39}$$

for all $a, b \in A$, $i, j \in I$;

(4) there exist elements $v_{ij} \in A$, $i, j \in I$, such that for all θ_i,

$$\Delta_A^R(\theta_i) = \sum_{j\in I} \theta_j \otimes v_{ji} \tag{40}$$

$$\Delta(v_{ik}) = \sum_{j\in I} v_{ij} \otimes v_{jk} \qquad \epsilon(v_{ik}) = \delta_{ik} \tag{41}$$

(5) for all $a \in A$,

$$\sum_{j\in I} v_{ji}(a \star f_{jk}) = \sum_{j\in I} (f_{ij} \star a) v_{kj} \tag{42}$$

where $a \star \alpha = \alpha(a_{(1)})a_{(2)}$ for all $\alpha \in A^*$, $a \in A$.

Conversely, if $(\theta_i)_{i\in I}$ is a basis of a vector space V, and we have functionals $(f_{ij})_{i,j\in I}$ defined on A and elements $(v_{ij})_{i,j\in I}$ in A which satisfy (39), (41) and (42), then there exists a unique bicovariant bimodule such that $V = \Gamma_{\mathrm{inv}}$ and (38) and (40) are satisfied.

This result places significant constraints on the possible bicovariant calculi which are consistent with our assumption that the differentials of the generators should generate the bimodule of forms as a left A-module. That assumption implies immediately that Γ_{inv} is spanned by the *finite* set of elements θ_{ik}. Choosing a basis from this set we either take all n^2 as linear independent elements, in which case the v_{ij}s of (40) and (41) have already been determined in (36), or we introduce linear relations between the θ_{ik}s. In the latter case we must be sure that such relations do not destroy the bicovariance—they must factor through the coactions. This condition will tend to fix the possible bases of Γ_{inv}, which in turn determines the v_{ij}s through (40) and (36). It then follows immediately that (41) is satisfied. Now, trying to introduce commutation relations which factor through the left and right coactions which we wish to maintain, it is not too difficult to see that when these relations take the form (38) with the f_{ij}s satisfying (39) and (42), bicovariance is maintained. Moreover, as the theorem states, the commutation relations in *any* bicovariant bimodule must *always* take this form. Thus our method of attacking the classification problem can now be discerned. We decide upon a valid basis of Γ_{inv}, and then assume the general form for the commutation relations, (38). We must then impose as constraints, consistency with the relations already present from the Leibniz rule, together with (39) and (42). This procedure will be made absolutely explicit for the case of the quantum groups of particular interest to us in the following section.

Remark 3.13. Having chosen a basis $(\theta_i)_{i\in I}$ for Γ_{inv}, the right-invariant elements $(\eta_i)_{i\in I}$ defined by $\eta_i = \sum_{j\in I} \theta_j S(v_{ji})$ form a basis for the vector space of right-invariant one-forms, Γ_{inv}.

Remark 3.14. The *dimension* of a bicovariant calculus is defined to be $\dim \Gamma_{\mathrm{inv}}$. We shall only be interested in finite, d-dimensional, examples so we will eschew the index set and consider indices running over a finite set.

Given two bicovariant bimodules, $(\Gamma, \Delta_A^L, \Delta_A^R)$ and $(\tilde{\Gamma}, \tilde{\Delta}_A^L, \tilde{\Delta}_A^R)$, of dimensions d and $\tilde{d}$ respectively, their tensor product over A, $(\Gamma' = \Gamma \otimes_A \tilde{\Gamma}, \Delta'^L_A, \Delta'^R_A)$ is also a bicovariant bimodule as follows. The left and right coactions of A on Γ' are given by

$$\Delta'^L_A(\omega \otimes \tilde{\omega}) = \omega_{(A)}\tilde{\omega}_{(A)} \otimes \omega_{(\Gamma)} \otimes \tilde{\omega}_{(\Gamma)} \qquad \Delta'^R_A(\omega \otimes \tilde{\omega}) = \omega_{(\Gamma)} \otimes \tilde{\omega}_{(\Gamma)} \otimes \omega_{(A)}\tilde{\omega}_{(A)} \tag{43}$$

and the f functionals of theorem 3.12 are now given by $F'_{ik,jl} = f_{ij} * \tilde{f}_{kl}$, where $*$ is the usual convolution product, such that

$$(\theta_i \otimes \tilde{\theta}_k)a = \sum_{j=1\ldots d, l=1\ldots \tilde{d}} (F'_{ik,jl} \star a)(\theta_j \otimes \tilde{\theta}_l) \tag{44}$$

for all $a \in A$, where $(\theta_i)_{i=1\ldots d}$ and $(\tilde{\theta}_i)_{i=1\ldots \tilde{d}}$ are the bases of left-invariant elements in Γ_{inv} and $\tilde{\Gamma}_{\mathrm{inv}}$ respectively, so that $(\theta_i \otimes \tilde{\theta}_j)_{i=1\ldots d, j=1\ldots \tilde{d}}$ is the basis of left-invariant elements in Γ'_{inv}. It is clear that in this way we can build arbitrary n-fold tensor powers, $\Gamma^{\otimes n} = \Gamma \otimes_A \Gamma \otimes_A \ldots \otimes_A \Gamma$ of a given bicovariant bimodule, all of which are themselves bicovariant with left and right coactions denoted $\Delta^{n}{}_A^L$ and $\Delta^{n}{}_A^R$ respectively. We can then define $\Gamma^{\otimes} = A \oplus \Gamma \oplus \Gamma^{\otimes 2} \oplus \ldots$ to be the analogue of the classical algebra of covariant tensor fields. $\Gamma^{\otimes}$ is a bicovariant graded algebra, in that it is a tensor algebra and a bicovariant bimodule over A, with coactions $\Delta^{\otimes}{}_A^L$ and $\Delta^{\otimes}{}_A^R$ which are algebra maps and coincide on elements of $\Gamma^{\otimes n}$ with the coactions $\Delta^{n}{}_A^L$ and $\Delta^{n}{}_A^R$ respectively.

The next step in Woronowicz's construction of a noncommutative geometry for Hopf algebras is to introduce an analogue of the classical external algebra of forms. Starting from the bicovariant graded algebra $\Gamma^{\otimes}$ we want to obtain another bicovariant graded algebra, the *external bicovariant graded algebra*, Ω, as a quotient, $\Omega = \Gamma^{\otimes}/S$, by some graded two-sided ideal S. Woronowicz introduces the *unique* linear bimodule map $\Lambda : \Gamma \otimes \Gamma \to \Gamma \otimes \Gamma$ such that

$$\Lambda(\theta \otimes \eta) = \eta \otimes \theta \tag{45}$$

for any $\theta \in \Gamma_{\mathrm{inv}}$, $\eta \in \Gamma_{\mathrm{inv}}$. Λ is then also a bicomodule map, is given explicitly on the basis of left-invariant elements $(\theta_i \otimes \theta_k)_{i,j=1\ldots d}$ by

$$\Lambda(\theta_i \otimes \theta_k) = \sum_{s,t=1\ldots d} \Lambda_{ik,st}\theta_s \otimes \theta_t = \sum_{s,t=1\ldots d} f_{it}(v_{sk})\theta_s \otimes \theta_t \tag{46}$$

and can be shown to satisfy the braid equation, $\Lambda_{12}\Lambda_{23}\Lambda_{12} = \Lambda_{23}\Lambda_{12}\Lambda_{23}$. The map Λ may then be used in just the same way as the permutation operator is used classically. Thus we define an analogue of the antisymmetrization operator on $\Gamma^{\otimes n}$, $W_{1\ldots n}$, by replacing the classical permutation operator by Λ everywhere in the classical antisymmetrizer. $W_{1\ldots n}$ is then a bimodule and bicomodule map $W_{1\ldots n} : \Gamma^{\otimes n} \to \Gamma^{\otimes n}$ and we can define $S^n = \ker W_{1\ldots n}$ so that $\Gamma^{\otimes n}/S^n$ is isomorphic to $\mathrm{Im}\, W_{1\ldots n}$ and $\Omega = \Gamma^{\otimes}/S$ where $S = \bigoplus_{n=2} S^n$. As $W_{1\ldots n}$ is a bicomodule map, the left and right coactions of $\Gamma^{\otimes}$ descend to the quotient where they shall be denoted $\Delta^{\Omega}{}_A^L$ and $\Delta^{\Omega}{}_A^R$ respectively. Moreover, we can now define the *wedge product* as $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n = W_{1\ldots n}(\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n)$ where the ω_i are arbitrary elements of Γ.

Remark 3.15. If we were to construct the bicovariant graded algebra $\tilde{\Gamma}^{\otimes}$ from a bicovariant bimodule $\tilde{\Gamma}$ which contains Γ as a sub-bimodule and a sub-bicomodule, then the bicovariant graded algebra $\Gamma^{\otimes}$ embeds naturally into $\tilde{\Gamma}^{\otimes}$ in the sense that there is the obvious natural embedding map $\phi : \Gamma^{\otimes} \to \tilde{\Gamma}^{\otimes}$ such that

$$\phi|_A = \mathrm{id} \qquad \phi|_\Gamma = \iota \tag{47}$$

$$\phi(\omega\rho) = \phi(\omega)\phi(\rho) \tag{48}$$

$$(\mathrm{id} \otimes \phi) \circ \Delta_A^L = \Delta^{\otimes}{}_A^L \circ \phi \qquad (\phi \otimes \mathrm{id}) \circ \Delta_A^R = \Delta^{\otimes}{}_A^R \circ \phi \tag{49}$$

where $\iota : \Gamma \to \tilde{\Gamma}$ is the natural inclusion. An attractive feature of the external bicovariant algebra construction, which has its origin in the uniqueness of the map Λ, is that this embedding survives the quotienting procedures so that $(\Omega, \Delta^{\Omega}{}_A^L, \Delta^{\Omega}{}_A^R)$ embeds naturally in $(\tilde{\Omega}, \Delta^{\tilde{\Omega}}{}_A^L, \Delta^{\tilde{\Omega}}{}_A^R)$.

We may now state the following theorem of Woronowicz.

Theorem 3.16. Let Ω be the external bicovariant algebra constructed above. There exists one and only one linear map $\mathrm{d} : \Omega \to \Omega$ such that:

(1) d increases the grade by one;

(2) on elements of grade 0, d coincides with the differential of the first-order bicovariant calculus $(\Gamma, \mathrm{d}, \Delta_A^L, \Delta_A^R)$;

(3) for all $\omega \in \Gamma^{\otimes k}$, $k = 0, 1, 2, \ldots$ and $\omega' \in \Omega$,

$$\mathrm{d}(\omega \wedge \omega') = \mathrm{d}\omega \wedge \omega' + (-1)^k \omega \wedge \mathrm{d}\omega' \tag{50}$$

(4) for any $\omega \in \Omega$,

$$\mathrm{d}(\mathrm{d}\omega) = 0 \tag{51}$$

(5) d is a both a left and a right comodule map i.e.

$$\Delta^{\Omega}{}_A^L(\mathrm{d}\omega) = (\mathrm{id} \otimes \mathrm{d}) \circ \Delta^{\Omega}{}_A^L(\omega) \tag{52}$$

$$\Delta^{\Omega}{}_A^R(\mathrm{d}\omega) = (\mathrm{d} \otimes \mathrm{id}) \circ \Delta^{\Omega}{}_A^R(\omega) \tag{53}$$

for all $\omega \in \Omega$.

Remark 3.17. The external bicovariant algebra Ω equipped with the differential described in this result will be called the *exterior bicovariant differential calculus* over the Hopf algebra A, and denoted $(\Omega, \mathrm{d}, \Delta^{\Omega}{}_A^L, \Delta^{\Omega}{}_A^R)$. Brzeziński has shown that this is a super-Hopf algebra [23].

Remark 3.18. To prove this theorem Woronowicz extends the bimodule Γ to $\tilde{\Gamma} = Ax \oplus \Gamma$ where Ax is the left A-module freely generated by the single element x, such that the right action of A on an element ax is given by $axb = abx + a\mathrm{d}b$ and the coactions are such that the element x is bi-invariant. The theorem is then proved for the external bicovariant algebra $\tilde{\Omega}$ built from $\tilde{\Gamma}$ with the final result for Ω coming after using the natural embedding of Ω in $\tilde{\Omega}$. Along the way the differential is expressed as $\mathrm{d}a = [x, a]$, for any $a \in A$, and more generally as $\mathrm{d}\omega = [x, \omega]_{\mp}$ for any $\omega \in \tilde{\Gamma}$ where $[x, \omega]_{\mp} = x \wedge \omega \mp \omega \wedge x$ with $-$ and $+$ for ω of even and odd grade respectively. In particular examples of bicovariant differential calculi it sometimes happens that there is a bi-invariant element *within* the unextended calculus which implements the differential in this way. Such calculi are called *inner*.

The final component of the Woronowicz differential calculus is the analogue of the classical Lie algebra of tangent vectors at the identity. Classically this is isomorphic to $(\ker \epsilon / (\ker \epsilon)^2)^*$. In the abstract Hopf algebra settings it turns out that defining a space $\mathcal{L}$ by $\mathcal{L} = (\ker \epsilon / \mathcal{R})^*$ there is a unique bilinear map, $\langle , \rangle : \Gamma \times \mathcal{L} \to \mathbb{C}$, such that for all $a \in A$, $\omega \in \Gamma$ and $\chi \in \mathcal{L}$,

$$\langle a\omega, \chi \rangle = \epsilon(a) \langle \omega, \chi \rangle \qquad \langle \mathrm{d}a, \chi \rangle = \chi(a) \tag{54}$$

which is *non-degenerate* as a pairing between Γ_{inv} and T. Thus, we can introduce a basis $(\chi_i)_{i=1\ldots d}$ for $\mathcal{L}$ dual to $(\theta_i)_{i=1\ldots d}$. Also, it is not too difficult to see that for any $\omega \in \Gamma$ there is an element $a \in \ker \epsilon$ such that $P(\omega) = P(\mathrm{d}a)$. Combined with the defining characteristics of the pairing between Γ and $\mathcal{L}$, we then arrive at a basis a_i of $(\ker \epsilon / \mathcal{R})$ such that $\theta_i = P(\mathrm{d}a_i)$. These a_i are analogues of the classical coordinate functions at the identity. The following result of Woronowicz continues the analogy with the classical situation.

Theorem 3.19. For all $a, b \in A$,

$$\mathrm{d}a = \sum_{j=1...d} (\chi_j \star a)\theta_j \tag{55}$$

$$\mathrm{d}\theta_i = -\sum_{j,l=1...d} \mathcal{C}_{jl,i}\theta_j \wedge \theta_l \tag{56}$$

$$\chi_i(ab) = \sum_{j=1...d} \chi_j(a) f_{ji}(b) + \epsilon(a)\chi_i(b) \tag{57}$$

$$\sum_{j=1...d} \chi_j(a) v_{ij} = \chi_i(a_{(2)}) S(a_{(1)}) a_{(3)} \tag{58}$$

where the f_{ji} and v_{ij} are as introduced in theorem 3.12, and $\mathcal{C}_{jl,i} = (\chi_j * \chi_l)(a_i)$.

Equation (57) shows us that the χ_i may be interpreted as 'deformed derivations' and is equivalent to a coproduct for the 'quantum tangent vectors',

$$\Delta(\chi_i) = \sum_{j=1...d} \chi_j \otimes f_{ji} + 1 \otimes \chi_i. \tag{59}$$

Further, as $\mathcal{R}$ is stable under the right-adjoint coaction, it follows that $\mathcal{L}$ is stable under the right-adjoint A^*-action, $\overset{\mathrm{ad}}{\triangleleft}$, that is, $S(\alpha_{(1)})\chi\alpha_{(2)} \in \mathcal{L}$ for all $\alpha \in A^*$ and any $\chi \in \mathcal{L}$. In the classical case this action, restricted to the tangent space, provides the Lie bracket. So we may define a *quantum Lie bracket* as,

$$[\chi_i, \chi_k] = \chi_i \overset{\mathrm{ad}}{\triangleleft} \chi_k = \sum_{j=1...d} \mathcal{C}_{ik,j}\chi_j \tag{60}$$

where the $\mathcal{C}_{ik,j}$ are analogues of the classical Lie algebra structure constants and are still to be determined. But from the coproduct on the χ_i we can expand the right-adjoint A^*-action, to obtain,

$$[\chi_i, \chi_k] = \chi_i\chi_k - \sum_{s=1...d} \chi_s(\chi_i \overset{\mathrm{ad}}{\triangleleft} f_{sk}). \tag{61}$$

We can then use (58) to determine $\chi_i \overset{\mathrm{ad}}{\triangleleft} f_{sk}$ and obtain an expression for the bracket as a *quantum commutator*,

$$[\chi_i, \chi_k] = \chi_i\chi_k - \sum_{s,t=1...d} \Lambda_{st,ik}\chi_s\chi_t. \tag{62}$$

The structure constants, $\mathcal{C}_{ik,j}$ may now be determined by evaluating the right-hand sides of (60) and (62) on a_l and equating the results to obtain,

$$\mathcal{C}_{ik,j} = \mathcal{C}_{ik,j} - \sum_{s,t=1...d} \Lambda_{st,ik}\mathcal{C}_{st,j}. \tag{63}$$

There is a *quantum Jacobi identity* for the quantum Lie bracket given by

$$[\chi_i, [\chi_j, \chi_k]] = [[\chi_i, \chi_j], \chi_k] - \sum_{s,t=1...d} \Lambda_{st,jk}[[\chi_i, \chi_s], \chi_t]. \tag{64}$$

The *universal enveloping algebra* of the quantum Lie algebra $\mathcal{L}$ may be introduced as the quotient of the tensor algebra of $\mathcal{L}$ by the two-sided ideal generated by the elements $\chi_i\chi_k - \sum_{s,t=1...d} \Lambda_{st,ik}\chi_s\chi_t - \sum_{j=1...d} \mathcal{C}_{ik,j}\chi_j$.

4. The classification procedure

In this section we make explicit the procedure, already outlined in the previous section, for determining under certain assumptions all possible first-order bicovariant differential calculi for a given quantum group. It was first applied by Müller-Hoissen [22, 24] to the case of the standard two-parameter quantum group $GL_{q,p}(2)$. Some further results appeared in subsequent papers [25, 26], and in [27] it was applied to the standard one-parameter quantum group $GL_q(3)$. Here we apply this 'recipe' to the cases of $GL_{h,g}(2)$ and $SL_h(2)$.

Starting with our quantum group A, where A is either $GL_{h,g}(2)$ or $SL_h(2)$, we introduce the first-order differential calculus, in the first instance, as the free A-bimodule Γ_0, on the symbols $\{\mathrm{d}_0 a, \mathrm{d}_0 b, \mathrm{d}_0 c, \mathrm{d}_0 d\}$ with the differential $\mathrm{d}_0 : GL_{h,g}(2) \to \Gamma_0$ defined on any element of A by way of the Leibniz rule. Note that $\mathrm{d}_0\mathcal{D}^{-1} = -\mathcal{D}^{-1}\mathrm{d}_0\mathcal{D}\mathcal{D}^{-1}$. However, as already mentioned, we pass directly to some quotient (Γ, d) which we assume is generated as a left A-module by $\{\mathrm{d}a, \mathrm{d}b, \mathrm{d}c, \mathrm{d}d\}$ and is still a bicovariant bimodule, denoted $(\Gamma, \mathrm{d}, \Delta_A^L, \Delta_A^R)$. Then, by remark 3.10 Γ_{inv} is spanned by the four left-invariant forms $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ where,

$$\begin{aligned} \theta_1 &= P(\mathrm{d}a) = S(a)\mathrm{d}a + S(b)\mathrm{d}c \qquad & \theta_2 &= P(\mathrm{d}b) = S(a)\mathrm{d}b + S(b)\mathrm{d}d \\ \theta_3 &= P(\mathrm{d}c) = S(c)\mathrm{d}a + S(d)\mathrm{d}c \qquad & \theta_4 &= P(\mathrm{d}d) = S(c)\mathrm{d}b + S(d)\mathrm{d}d. \end{aligned} \tag{65}$$

In the other direction, from (34), the differentials of the generators may be written in terms of the left invariant θ_is as,

$$\begin{aligned} \mathrm{d}a &= a_{(1)}P(\mathrm{d}a_{(2)}) = a\theta_1 + b\theta_3 \qquad & \mathrm{d}b &= b_{(1)}P(\mathrm{d}b_{(2)}) = a\theta_2 + b\theta_4 \\ \mathrm{d}c &= c_{(1)}P(\mathrm{d}c_{(2)}) = c\theta_1 + d\theta_3 \qquad & \mathrm{d}d &= d_{(1)}P(\mathrm{d}d_{(2)}) = c\theta_2 + d\theta_4. \end{aligned} \tag{66}$$

We are now free to choose from, $\{\theta_1, \theta_2, \theta_3, \theta_4\}$, a basis for Γ_{inv} and look for corresponding nontrivial first-order bicovariant differential calculi. We choose to look for calculi with the classical dimension, so for $GL_{h,g}(2)$ we make the further assumption that $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ is a basis of Γ_{inv} and call any extant calculi four-dimensional calculi, while for $SL_h(2)$ we look for three-dimensional calculi by assuming that $\{\theta_1, \theta_2, \theta_3\}$ is a basis of G_{inv} with $\theta_4 = \alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\theta_3$ and the coefficients α_1, α_2 and α_3 to be determined. The assumptions will be justified if we find nontrivial calculi. This in turn will be established if we can find functionals f_{ij} and elements v_{ij} as in theorem 3.12 consistent with our assumptions.

As we discussed in the previous section, the elements v_{ij} are already fixed through our assumptions, by (27), (65) and (40). For the four-dimensional, $GL_{h,g}(2)$, calculi, we have

$$V = ||v_{ij}|| = \begin{pmatrix} S(a)a & S(a)b & S(c)a & S(c)b \\ S(a)c & S(a)d & S(c)c & S(c)d \\ S(b)a & S(b)b & S(d)a & S(d)b \\ S(b)c & S(b)d & S(d)c & S(d)d \end{pmatrix}. \tag{67}$$

For the three-dimensional, $SL_h(2)$, calculi we have

$$V = \|v_{ij}\| = \begin{pmatrix} S(a)a + \alpha_1 S(b)c & S(a)b + \alpha_1 S(b)d & S(c)a + \alpha_1 S(d)c \\ S(a)c + \alpha_2 S(b)c & S(a)d + \alpha_2 S(b)d & S(c)c + \alpha_2 S(d)c \\ S(b)a + \alpha_3 S(b)c & S(b)b + \alpha_3 S(b)d & S(d)a + \alpha_3 S(d)c \end{pmatrix} \tag{68}$$

but in this case we must also have,

$$\Delta_A^R(\theta_4 - \alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\theta_3) = 0. \tag{69}$$

Evaluating (69) with (65), (27) and (66) fixes the α_i coefficients to be given by $\alpha_1 = -1$, $\alpha_2 = 0$ and $\alpha_3 = -2h$, so that we must have

$$\theta_4 = -\theta_1 - 2h\theta_3. \tag{70}$$

In fact, when we look for bi-invariant forms in the four-dimensional calculi, we soon find that up to scalar multiplication there is but one, which we denote by $\mathrm{Tr}_h\,\Theta$, and which is given by

$$\mathrm{Tr}_h\,\Theta = \theta_1 + 2h\theta_3 + \theta_4. \tag{71}$$

Existence of the bicovariant first-order differential calculi which we seek, now hinges entirely on the f_{ij}s. Following Müller-Hoissen [22], let us refine our notation slightly. For the four-dimensional calculi, we write relations (38) in terms of four 4×4 matrices, the '$ABCD$' matrices, $A_{ij} = f_{ij}(a)$, $B_{ij} = f_{ij}(b)$, $C_{ij} = f_{ij}(c)$ and $D_{ij} = f_{ij}(d)$, as

$$\begin{aligned} \theta_i a &= \sum_{j=1\ldots4} (aA_{ij} + bC_{ij})\theta_j \qquad & \theta_i b &= \sum_{j=1\ldots4} (aB_{ij} + bD_{ij})\theta_j \\ \theta_i c &= \sum_{j=1\ldots4} (cA_{ij} + dC_{ij})t_j \qquad & \theta_i d &= \sum_{j=1\ldots4} (cB_{ij} + dD_{ij})\theta_j. \end{aligned} \tag{72}$$

In the case of the three-dimensional calculi the 4×4 $ABCD$ matrices are simply replaced by 3×3 $ABCD$ matrices with the summations then to 3. We may now list the constraints on the $ABCD$ matrices.

Constraint 1. Differentiating the quantum group relations, (5), to obtain, in R-matrix form,

$$\mathrm{d}(RT_1T_2 - T_2T_1R) = R(\mathrm{d}T_1)T_2 + RT_1(\mathrm{d}T_2) - (\mathrm{d}T_2)T_1R - T_2(\mathrm{d}T_1)R = 0 \tag{73}$$

we replace the differentials by left-invariant forms through (66). We then use (72) to commute the $\theta_i s$ to the right which then allows us to equate the (ordered) algebra valued coefficients and obtain linear relations between the matrix elements of the $ABCD$ matrices. Similarly for the three-dimensional case, but then, when replacing the differentials by left-invariant forms, we also use (70).

Constraint 2. In both the three- and four-dimensional cases relations (42) may be expressed in the following matrix form,

$$\begin{pmatrix} V^TA & V^TB \\ V^TC & V^TD \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} AV^T & BV^T \\ CV^T & DV^T \end{pmatrix}. \tag{74}$$

Only algebra elements appear in these equations and once all terms are 'straightened' they yield further linear relations among the matrix elements of the $ABCD$ matrices.

Constraint 3. Equations (39) tell us that A, B, C and D must be the representation matrices of a, b, c and d respectively, and that the matrix representation of the determinant, $\mathbb{D}$ say, where $\mathbb{D} = AD - BC + hAC$, must be invertible in the case of $GL_{h,g}(2)$, and equal to the identity in the case of $SL_h(2)$. Imposing these conditions we obtain nonlinear relations amongst the $ABCD$ matrix elements.

Remark 4.1. Recall from remark 3.2 our definition of isomorphism for differential calculi. As we assume a single θ_i basis, different possible $ABCD$ matrices must correspond to nonisomorphic calculi.

The $ABCD$ matrices which result from this procedure provide the most general possible first-order bicovariant calculi under the stated assumptions. We may now investigate the external bicovariant graded algebras, and also the 'quantum Lie algebras' which are related to our first-order calculi.

We begin by using theorem 3.16, in particular (51), together with (66) to deduce the structure constants $\mathcal{C}_{ij,k}$ appearing in the 'Cartan–Maurer equations', (56). For the four-dimensional calculi, we obtain four 4×4 matrices,

$$\mathcal{C}_{ij,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{C}_{ij,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\mathcal{C}_{ij,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \mathcal{C}_{ij,4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{75}$$

while for the three-dimensional calculi we obtain three 3×3 matrices,

$$\mathcal{C}_{ij,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{C}_{ij,2} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -2h \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{C}_{ij,3} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -2h \end{pmatrix} \tag{76}$$

together with a relation,

$$2\theta_1 \wedge \theta_1 + 4h\theta_3 \wedge \theta_1 + \theta_2 \wedge \theta_3 + \theta_3 \wedge \theta_2 = 0 \tag{77}$$

which will have to be consistent with any commutation relations which we derive for left-invariant forms of the three-dimensional calculi. In fact these commutation relations can now be obtained by differentiating the relations (72) using (50). For the four-dimensional calculi we obtain the following four sets of equations,

$$\begin{aligned}
&(\mathcal{C}_{st,j}A_{ij} - \mathcal{C}_{jk,i}(A_{js}A_{kt} + B_{js}C_{kt}))\theta_s \wedge \theta_t \\
&\qquad = A_{ij}(\theta_1 \wedge \theta_j + \theta_j \wedge \theta_1) + B_{ij}\theta_j \wedge \theta_3 + C_{ij}\theta_2 \wedge \theta_j \\
&(\mathcal{C}_{st,j}C_{ij} - \mathcal{C}_{jk,i}(C_{js}A_{kt} + D_{js}C_{kt}))\theta_s \wedge \theta_t \\
&\qquad = C_{ij}(\theta_4 \wedge \theta_j + \theta_j \wedge \theta_1) + D_{ij}\theta_j \wedge \theta_3 + A_{ij}\theta_3 \wedge \theta_j \\
&(\mathcal{C}_{st,j}B_{ij} - \mathcal{C}_{jk,i}(A_{js}B_{kt} + B_{js}D_{kt}))\theta_s \wedge \theta_t \\
&\qquad = B_{ij}(\theta_1 \wedge \theta_j + \theta_j \wedge \theta_4) + A_{ij}\theta_j \wedge \theta_2 + D_{ij}\theta_2 \wedge \theta_j \\
&(\mathcal{C}_{st,j}D_{ij} - \mathcal{C}_{jk,i}(C_{js}B_{kt} + D_{js}D_{kt}))\theta_s \wedge \theta_t \\
&\qquad = D_{ij}(\theta_4 \wedge \theta_j + \theta_j \wedge \theta_4) + C_{ij}\theta_j \wedge \theta_2 + B_{ij}\theta_3 \wedge \theta_j
\end{aligned} \tag{78}$$

where repeated indices are summed from 1 to 4. In the three-dimensional case the commutation relations are given by,

$$\begin{aligned}
&(\mathcal{C}_{st,j}A_{ij} - \mathcal{C}_{jk,i}(A_{js}A_{kt} + B_{js}C_{kt}))\theta_s \wedge \theta_t \\
&\qquad = A_{ij}(\theta_1 \wedge \theta_j + \theta_j \wedge \theta_1) + B_{ij}\theta_j \wedge \theta_3 + C_{ij}\theta_2 \wedge \theta_j \\
&(\mathcal{C}_{st,j}C_{ij} - \mathcal{C}_{jk,i}(C_{js}A_{kt} + D_{js}C_{kt}))\theta_s \wedge \theta_t \\
&\qquad = C_{ij}(\theta_j \wedge \theta_1 - \theta_1 \wedge \theta_j - 2h\theta_3 \wedge \theta_j) + D_{ij}\theta_j \wedge \theta_3 + A_{ij}\theta_3 \wedge \theta_j \\
&(\mathcal{C}_{st,j}B_{ij} - \mathcal{C}_{jk,i}(A_{js}B_{kt} + B_{js}D_{kt}))\theta_s \wedge \theta_t \\
&\qquad = B_{ij}(\theta_1 \wedge \theta_j - \theta_j \wedge \theta_1 - 2h\theta_j \wedge \theta_3) + A_{ij}\theta_j \wedge \theta_2 + D_{ij}\theta_2 \wedge \theta_j \\
&(\mathcal{C}_{st,j}D_{ij} - \mathcal{C}_{jk,i}(C_{js}B_{kt} + D_{js}D_{kt})\theta_s \wedge \theta_t \\
&\qquad = -D_{ij}(\theta_1 \wedge \theta_j + \theta_j \wedge \theta_1 + 2h\theta_3 \wedge \theta_j + 2h\theta_j \wedge \theta_3) + C_{ij}\theta_j \wedge \theta_2 + B_{ij}\theta_3 \wedge \theta_j
\end{aligned} \tag{79}$$

where repeated indices are now summed from 1 to 3.

Recalling that $\mathrm{Tr}_h\,\Theta$ was the single bi-invariant form in the four-dimensional calculi, and also the defining characteristic of Woronowicz's bimodule map Λ, (45), we obtain a general relation which must be consistent with commutation relations between one-forms in the four-dimensional calculi,

$$\begin{aligned}0 = \mathrm{Tr}_h\,\Theta \wedge \mathrm{Tr}_h\,\Theta = \theta_1 \wedge \theta_1 + 4h^2\theta_3 \wedge \theta_3 + \theta_4 \wedge \theta_4 + 2h(\theta_1 \wedge \theta_3 + \theta_3 \wedge \theta_1) + \theta_1 \wedge \theta_4 \\ +\theta_4 \wedge \theta_1 + 2h(\theta_3 \wedge \theta_4 + \theta_4 \wedge \theta_3).\end{aligned} \tag{80}$$

Further, in any calculi where this bi-invariant form implements the differential in the sense of remark (3.18), so that on arbitrary one-forms ω we have,

$$\mathrm{d}\omega = \frac{1}{\kappa}[\mathrm{Tr}_h\,\Theta, \omega]_+ \tag{81}$$

where κ is some constant, we have further relations,

$$[\mathrm{Tr}_h\,\Theta, \theta_i] = \kappa \sum_{j,k=1\ldots4} \mathcal{C}_{jk,i}\theta_j \wedge \theta_k. \tag{82}$$

Again these must be consistent with relations coming from (79).

The commutator expression for the quantum Lie bracket, (62), requires that we know the explicit form of the matrix Λ whose components are given in (46) as $\Lambda_{ij,st} = f_{it}(v_{sk})$. But we know the algebraic elements of the matrix $||v_{ij}||$ and therefore their expressions in terms of the $ABCD$ representation. This is all that is required.

Finally, the structure constants, $\mathcal{C}_{ij,k}$, for the quantum Lie bracket, (60), now follow immediately from (63) as we know the $\mathcal{C}_{ij,k}$s and $\Lambda_{ij,st}$s.

5. Four-dimensional bicovariant calculi on $GL_{h,g}(2)$

We summarize the result of applying the procedure of the previous section to $GL_{h,g}(2)$ in the following theorems.

Theorem 5.1. There are three one-parameter families of four-dimensional first-order bicovariant differential calculi on $GL_{h,g}(2)$ whose bimodules of forms are generated as left $GL_{h,g}(2)$-modules by the differentials of the quantum group generators. We denote the three families by $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$. They are completely characterized by their respective $ABCD$ matrices.

$\Gamma_1^{4\mathrm{D}}$: The $ABCD$ matrices are given by,

$$\begin{aligned}
A &= \begin{pmatrix} \frac{3z+2}{2} & 0 & \frac{-(3h+g)z-2h}{2} & \frac{-z}{2} \\ h(z+1) & z+1 & -h^2(z+1) & -h(z+1) \\ 0 & 0 & z+1 & 0 \\ \frac{z}{2} & 0 & \frac{(h-g)z+2h}{2} & \frac{z+2}{2} \end{pmatrix} \\
B &= \begin{pmatrix} 0 & z & h(h+g)(z+1) & 0 \\ 0 & (h+g)(z+1) & -hg(h+g)(z+1) & 0 \\ 0 & 0 & -(h+g)(z+1) & 0 \\ 0 & z & g(h+g)(z+1) & 0 \end{pmatrix} \\
C &= \begin{pmatrix} 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 \end{pmatrix}
\end{aligned} \tag{83}$$

$$D = \begin{pmatrix} \frac{z+2}{2} & 0 & \frac{-(h-g)z+2g}{2} & \frac{z}{2} \\ -g(z+1) & z+1 & -g^2(z+1) & g(z+1) \\ 0 & 0 & z+1 & 0 \\ \frac{-z}{2} & 0 & \frac{-(h+3g)z-2g}{2} & \frac{3z+2}{2} \end{pmatrix}.$$

Here we must have $z \neq -1$ to ensure the invertibility of $\mathbb{D}$. With $g = h$, the quantum determinant is central in the differential calculus for parameter values $z = 0$ and $z = -2$. The differential of the quantum determinant is

$$\mathrm{d}\mathcal{D} = \frac{z+2}{2}\mathcal{D}\,\mathrm{Tr}_h\,\Theta \tag{84}$$

and for $z \neq 0$ the calculi are inner,

$$\begin{aligned} \mathrm{d}a &= \frac{1}{2z}[\mathrm{Tr}_h\,\Theta, a] \qquad & \mathrm{d}b &= \frac{1}{2z}[\mathrm{Tr}_h\,\Theta, b] \\ \mathrm{d}c &= \frac{1}{2z}[\mathrm{Tr}_h\,\Theta, c] \qquad & \mathrm{d}d &= \frac{1}{2z}[\mathrm{Tr}_h\,\Theta, d]. \end{aligned} \tag{85}$$

$\Gamma_2^{4\mathrm{D}}$: The $ABCD$ matrices are given by,

$$\begin{aligned} A &= \begin{pmatrix} \frac{z+2}{2} & 0 & \frac{(h+g)z-2h}{2} & \frac{z}{2} \\ h(z+1) & 1 & h((h+g)z-h) & h(z-1) \\ 0 & 0 & 1 & 0 \\ \frac{-z}{2} & 0 & \frac{-(h+g)z+2h}{2} & \frac{-z+2}{2} \end{pmatrix} \\ B &= \begin{pmatrix} -hz & 0 & h(h+g)(1-z) & -hz \\ hgz & (h+g) & hg(h+g)(z-1) & ghz \\ z & 0 & (h+g)(z-1) & z \\ -gz & 0 & g(h+g)(1-z) & -gz \end{pmatrix} \\ C &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ z & 0 & (h+g)z & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ D &= \begin{pmatrix} \frac{-z+2}{2} & 0 & \frac{-(h+g)z+2g}{2} & \frac{-z}{2} \\ g(z-1) & 1 & g((h+g)z-g) & g(z+1) \\ 0 & 0 & 1 & 0 \\ \frac{z}{2} & 0 & \frac{(h+g)z-2g}{2} & \frac{z+2}{2} \end{pmatrix}. \end{aligned} \tag{86}$$

There is no restriction on the value of z in this case. With $g = h$, the quantum determinant is central in the differential calculi for all values of z. The differential of the quantum determinant is

$$\mathrm{d}\mathcal{D} = \frac{2-3z}{2}\mathcal{D}\,\mathrm{Tr}_h\,\Theta \tag{87}$$

but the calculi are not inner,

$$\begin{aligned} &[\mathrm{Tr}_h\,\Theta, a] = 0 \qquad & &[\mathrm{Tr}_h\,\Theta, b] = 0 \\ &[\mathrm{Tr}_h\,\Theta, c] = 0 \qquad & &[\mathrm{Tr}_h\,\Theta, d] = 0. \end{aligned} \tag{88}$$

Γ_3^{4D}: The $ABCD$ matrices are given by,

$$A = \begin{pmatrix} \frac{z+2}{2} & 0 & \frac{(h+g)z-2h}{2} & \frac{z}{2} \\ h & 1 & -h^2 & -h \\ 0 & 0 & 1 & 0 \\ \frac{z}{2} & 0 & \frac{(h+g)z+2h}{2} & \frac{z+2}{2} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & h(h+g) & 0 \\ 0 & (h+g) & -hg(h+g) & 0 \\ 0 & 0 & -(h+g) & 0 \\ 0 & 0 & g(h+g) & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} \frac{z+2}{2} & 0 & \frac{(h+g)z+2g}{2} & \frac{z}{2} \\ -g & 1 & -g^2 & g \\ 0 & 0 & 1 & 0 \\ \frac{z}{2} & 0 & \frac{(h+g)z-2g}{2} & \frac{z+2}{2} \end{pmatrix}. \tag{89}$$

Like Γ_1^{4D} we must have $z \neq -1$ to ensure the invertibility of $\mathbb{D}$. With $g = h$, the quantum determinant is central in the differential calculus for parameter values $z = 0$ and $z = -2$. The differential of the quantum determinant is

$$\mathrm{d}\mathcal{D} = \frac{z+2}{2}\mathcal{D}\,\mathrm{Tr}_h\,\Theta \tag{90}$$

and again the calculi are not inner,

$$\begin{aligned} [\mathrm{Tr}_h\,\Theta, a] &= za\,\mathrm{Tr}_h\,\Theta \qquad & [\mathrm{Tr}_h\,\Theta, b] &= zb\,\mathrm{Tr}_h\,\Theta \\ [\mathrm{Tr}_h\,\Theta, c] &= zc\,\mathrm{Tr}_h\,\Theta \qquad & [\mathrm{Tr}_h\,\Theta, d] &= zd\,\mathrm{Tr}_h\,\Theta. \end{aligned} \tag{91}$$

For $z = 0$, $\Gamma_1^{4D} = \Gamma_2^{4D} = \Gamma_3^{4D}$, whilst for all other parameter values the calculi are distinct.

Proof. The matrices were obtained by systematically extracting the implications of constraints 1–3. This laborious task is made possible by using the computer algebra package REDUCE [28]. REDUCE was also used to check the other results. The results regarding the centrality of the quantum determinant were obtained by investigating under what conditions $\mathbb{D} = I$, since this is precisely the requirement imposed by relations (72). To obtain the expressions for the differential of the quantum determinant in the three calculi, we differentiate the expression $\mathcal{D} = ad - bc + hac$, replace on the right-hand side differentials by θ_is through (66), use the commutations relations between the θ_i and the algebra generators provided by the $ABCD$ matrices to commute the left-invariant forms θ_i to the right and finally straighten the algebra coefficients of the θ_i to obtain the quoted results. □

Remark 5.2. Müller-Hoissen [22] obtained a corresponding result for first-order differential calculi on the *standard* two-parameter quantum group, $GL_{q,p}(2)$. In that case, there is just a *single* one-parameter family of calculi, and *every* member of this family is inner.

Armed with the commutation relations between left-invariant forms and generators provided by the classification of $ABCD$ matrices, we may now obtain commutation relations between the differentials of the generators, and the generators. For example, in Γ_1^{4D}, starting with $\mathrm{d}aa$, we replace $\mathrm{d}a$ by its expression in terms of left-invariant forms, commute these to the right and then replace the left-invariant forms by their expressions in terms of differentials to obtain,

$$\begin{aligned} \mathrm{d}aa = (a + hc)\mathrm{d}a + \frac{z}{4}\{&9a + (12h - 7g)c + \mathcal{D}^{-1}\{bac - 3a^2d + 2(h-g) \\ &\times\{4bc^2 - 6adc - (h-2g)dc^2 + 3h(h-2g)c^3\}\}\}\mathrm{d}a \\ &+\frac{z}{2}\mathcal{D}^{-1}\{a^2c + (h-g)\{2ac^2 + (h-2g)c^3\}\}\mathrm{d}b \\ &+(ghc - ha)\mathrm{d}c + \frac{z}{4}\{(7g-6h)a - ((h-7g)(4h-3g) + gh)c \end{aligned}$$

$$
\begin{aligned}
&+\mathcal{D}^{-1}\{2ba^2 - 6ha^3 + 3(2h-3g)a^2d - (2h-3g)bac + 2(h-g)\\
&\times\{3(h-6g)adc - 2(h-6g)bc^2 - 3g(h-2g)dc^2 + 9gh(h-2g)c^3\}\}\}\mathrm{d}c\\
&+\frac{-z}{2}\mathcal{D}^{-1}\{a^3 + (2h-3g)a^2c + (h-g)\{(h-6g)ac^2 - 3g(h-2g)c^3\}\}\mathrm{d}d.
\end{aligned}
\tag{92}
$$

Most of the other commutation relations are much more complicated so we have chosen not to reproduce them here. A feature of this commutation relation, which is shared with the other differential-generator commutation relations for $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$, is that it is not quadratic, but that when $z = 0$, in which case $\Gamma_1^{4\mathrm{D}} = \Gamma_2^{4\mathrm{D}} = \Gamma_3^{4\mathrm{D}}$, it simplifies drastically and does indeed become quadratic. Moreover, when $z = 0$ these quadratic differential-generator commutation relations may be written in R-matrix form as†

$$\hat{R}^{-1}\mathrm{d}T_1T_2 = T_1\mathrm{d}T_2\hat{R}. \tag{93}$$

This R-matrix expression first appeared in the works of Schirrmacher [29] and Sudbery [30] which were developments on the works of Manin [13, 31] and Maltsiniotis [32]. These authors treated the coplane, in our case $\mathbb{A}_{-1}^{2|0}$, as the algebra of differentials $\mathrm{d}x_i$ of the 'coordinates' x_i whose algebra is that of the plane, $\mathbb{A}_1^{2|0}$. They then sought differential calculi expressed in terms of generators T_{ij} and their differentials $\mathrm{d}T_{ij}$ such that the plane and coplane are invariant under the transformations $x_i \mapsto \sum T_{ij}x_j$ and $\mathrm{d}x_i \mapsto \sum T_{ij}\mathrm{d}x_j + \sum \mathrm{d}T_{ij}x_j$. In the context of Jordanian quantum groups, in [33] the author *postulated* (93) as the relation defining commutation relations between differentials and generators for $SL_h(2)$.

Remark 5.3. In [26], where the differential calculus on the standard quantum group $GL_{q,p}(2)$ is considered, the authors also observe that the differential-generator commutation relations are not quadratic for general calculi in the one-parameter family, but that once their free parameter is fixed to zero, quadratic relations are obtained. Moreover in this case they also recover commutation relations with an R-matrix expression.

Remark 5.4. From (93) it is a simple matter to demonstrate that for the $z = 0$ calculus on $GL_{h,g}(2)$, the commutation relations between the left-invariant forms and the quantum group generators may be written as,

$$\Theta_1T_2 = T_2R_{21}\Theta_1R_{12} \tag{94}$$

where

$$\Theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}. \tag{95}$$

Incidentally, this is precisely the relation we obtain if we attempt a naive application of Jurco's construction with, in the notation of [41], $\Gamma = \Gamma_1^c \otimes_A \Gamma_1$.

Turning now to the higher-order calculi, we have the following result describing commutation relations between the left-invariant forms.

Theorem 5.5. The commutation relations between the left-invariant forms in the external bicovariant graded algebras $\Omega_1^{4\mathrm{D}}$, $\Omega_2^{4\mathrm{D}}$ and $\Omega_3^{4\mathrm{D}}$ built respectively upon the three families of first-order calculi, $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ using Woronowicz's theory as described in section 3 are as follows,

† Of course, $\hat{R}^{-1} = \hat{R}$, but we write it this way to make comparison with the general results of other authors explicit.

Ω_1^{4D}:

$$
\begin{aligned}
&\theta_3 \wedge \theta_3 = 0 \\
&\theta_3 \wedge \theta_4 = -\frac{2z+1}{z+1}\theta_4 \wedge \theta_3 + \frac{z}{z+1}\theta_1 \wedge \theta_3 \\
&\theta_3 \wedge \theta_1 = -\frac{1}{z+1}\theta_1 \wedge \theta_3 - \frac{z}{z+1}\theta_4 \wedge \theta_3 \\
&\theta_3 \wedge \theta_2 = -\theta_2 \wedge \theta_3 + (h+g)\theta_1 \wedge \theta_3 - (h+g)\theta_4 \wedge \theta_3 \\
&\theta_4 \wedge \theta_4 = \frac{z}{z+1}\theta_2 \wedge \theta_3 - \frac{z(h+g)}{z+1}\theta_1 \wedge \theta_3 + \frac{z(h+g)}{z+1}\theta_4 \wedge \theta_3 \\
&\theta_4 \wedge \theta_1 = -\theta_1 \wedge \theta_4 - \frac{z(h+g)}{z+1}\theta_1 \wedge \theta_3 + \frac{z(h+g)}{z+1}\theta_4 \wedge \theta_3 \\
&\theta_4 \wedge \theta_2 = -\frac{2z+1}{z+1}\theta_2 \wedge \theta_4 + \frac{z}{z+1}\theta_2 \wedge \theta_1 + \frac{(2z+1)(h+g)}{z+1}\theta_2 \wedge \theta_3 - (h+g)^2\theta_1 \wedge \theta_3 \\
&\qquad\qquad +(h+g)^2\theta_4 \wedge \theta_3 \\
&\theta_1 \wedge \theta_1 = -\frac{z}{z+1}\theta_2 \wedge \theta_3 \\
&\theta_1 \wedge \theta_2 = -\frac{1}{z+1}\theta_2 \wedge \theta_1 - \frac{z}{z+1}\theta_2 \wedge \theta_4 - \frac{(h+g)}{z+1}\theta_2 \wedge \theta_3 \\
&\theta_2 \wedge \theta_2 = (h+g)\theta_1 \wedge \theta_2 - (h+g)\theta_2 \wedge \theta_4 + (h+g)^2\theta_2 \wedge \theta_3
\end{aligned}
\tag{96}
$$

Ω_2^{4D}:

$$
\begin{aligned}
&\theta_3 \wedge \theta_3 = 0 \\
&\theta_3 \wedge \theta_4 = -\theta_4 \wedge \theta_3 \\
&\theta_3 \wedge \theta_1 = -\theta_1 \wedge \theta_3 \\
&\theta_3 \wedge \theta_2 = -\theta_2 \wedge \theta_3 + (h+g)\theta_1 \wedge \theta_3 - (h+g)\theta_4 \wedge \theta_3 \\
&\theta_4 \wedge \theta_4 = 0 \\
&\theta_4 \wedge \theta_1 = -\theta_1 \wedge \theta_4 \\
&\theta_4 \wedge \theta_2 = -\theta_2 \wedge \theta_4 + (h+g)\theta_2 \wedge \theta_3 - (h+g)^2\theta_1 \wedge \theta_3 + (h+g)^2\theta_4 \wedge \theta_3 \\
&\theta_1 \wedge \theta_1 = 0 \\
&\theta_1 \wedge \theta_2 = -\theta_2 \wedge \theta_1 - (h+g)\theta_2 \wedge \theta_3 \\
&\theta_2 \wedge \theta_2 = (h+g)\theta_1 \wedge \theta_2 - (h+g)\theta_2 \wedge \theta_4 + (h+g)^2\theta_2 \wedge \theta_3
\end{aligned}
\tag{97}
$$

Ω_3^{4D}:

$$
\begin{aligned}
&\theta_3 \wedge \theta_3 = 0 \\
&\theta_3 \wedge \theta_4 = -\theta_4 \wedge \theta_3 \\
&\theta_3 \wedge \theta_1 = -\theta_1 \wedge \theta_3 \\
&\theta_3 \wedge \theta_2 = -\theta_2 \wedge \theta_3 + (h+g)\theta_1 \wedge \theta_3 - (h+g)\theta_4 \wedge \theta_3 \\
&\theta_4 \wedge \theta_4 = 0 \\
&\theta_4 \wedge \theta_1 = -\theta_1 \wedge \theta_4 \\
&\theta_4 \wedge \theta_2 = -\theta_2 \wedge \theta_4 + (h+g)\theta_2 \wedge \theta_3 - (h+g)^2\theta_1 \wedge \theta_3 + (h+g)^2\theta_4 \wedge \theta_3 \\
&\theta_1 \wedge \theta_1 = 0 \\
&\theta_1 \wedge \theta_2 = -\theta_2 \wedge \theta_1 - (h+g)\theta_2 \wedge \theta_3 \\
&\theta_2 \wedge \theta_2 = (h+g)\theta_1 \wedge \theta_2 - (h+g)\theta_2 \wedge \theta_4 + (h+g)^2\theta_2 \wedge \theta_3.
\end{aligned}
\tag{98}
$$

The relations in each case are consistent with the respective relations (72). They are also consistent with relation (80) and in the case of $\Omega_1^{4\mathrm{D}}$, the relations (82). Further, they are such that $\{\theta_2^\alpha\theta_1^\beta\theta_4^\gamma\theta_3^\delta : \alpha, \beta, \gamma, \delta \in \{0, 1\}\}$ is a basis for the exterior algebra of forms in each case.

Proof. Once again we used REDUCE to check these results based on the discussion of section 4. The 16 equations (79) were treated, in each of the cases, $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ linear relations between the 10 'mis-ordered' two-forms, $\theta_3 \wedge \theta_3$, $\theta_3 \wedge \theta_4$, $\theta_3 \wedge \theta_1$, $\theta_3 \wedge \theta_2$, $\theta_4 \wedge \theta_4$, $\theta_4 \wedge \theta_1$, $\theta_4 \wedge \theta_2$, $\theta_1 \wedge \theta_1$, $\theta_1 \wedge \theta_2$ and $\theta_2 \wedge \theta_2$, and the six 'ordered' two-forms, $\theta_2 \wedge \theta_1$, $\theta_2 \wedge \theta_4$, $\theta_2 \wedge \theta_3$, $\theta_1 \wedge \theta_4$, $\theta_1 \wedge \theta_3$ and $\theta_4 \wedge \theta_3$. These linear relations were then solved for the 10 mis-ordered two-forms yielding in each case the single solution presented. Consistency with the relations (72) in each of the three cases was checked by commuting the generators through the relations and observing that no further conditions were incurred. The other consistency conditions were again checked by direct computation. Finally, observing that the relations are compatible with the ordering $\theta_2 \prec \theta_1 \prec \theta_4 \prec \theta_3$, we may use the diamond lemma to prove the statement about the bases. □

Remark 5.6. It is interesting to note here that the relations in $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ are the same and indeed could be obtained from those in $\Gamma_1^{4\mathrm{D}}$ by setting $z = 0$.

Remark 5.7. In contrast with the results in theorem 5.5, in the work of Müller-Hoissen and Reuten on $GL_{q,p}(2)$ [26] the corresponding commutation relations between left-invariant forms exhibited ordering circles which introduced further constraints on the free parameter of their family of calculi.

Let us now describe the quantum Lie brackets in the quantum Lie algebras $\mathcal{L}_1^{4\mathrm{D}}$, $\mathcal{L}_2^{4\mathrm{D}}$ and $\mathcal{L}_3^{4\mathrm{D}}$ dual to the bicovariant bimodules $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ respectively.

Theorem 5.8. The quantum Lie brackets and quantum commutators for the quantum Lie algebras $\mathcal{L}_1^{4\mathrm{D}}$, $\mathcal{L}_2^{4\mathrm{D}}$ and $\mathcal{L}_3^{4\mathrm{D}}$ as described in section 3 are as follows.

$\mathcal{L}_1^{4\mathrm{D}}$: The bracket relations are,

$$\begin{aligned}
&[\chi_1, \chi_1] = 0\\
&[\chi_1, \chi_2] = \frac{1}{z+1}\chi_2\\
&[\chi_1, \chi_3] = -\frac{1}{z+1}\chi_3 + \frac{h+g}{z+1}\chi_4\\
&[\chi_1, \chi_4] = 0\\
&[\chi_2, \chi_1] = -\frac{1}{z+1}\chi_2\\
&[\chi_2, \chi_2] = 0\\
&[\chi_2, \chi_3] = \frac{1}{z+1}\chi_1 - \frac{h+g}{z+1}\chi_2 - \frac{1}{z+1}\chi_4\\
&[\chi_2, \chi_4] = \frac{1}{z+1}\chi_2\\
&[\chi_3, \chi_1] = -\frac{h+g}{z+1}\chi_1 + \frac{1}{z+1}\chi_3\\
&[\chi_3, \chi_2] = -\frac{1}{z+1}\chi_1 - \frac{h+g}{z+1}\chi_2 + \frac{1}{z+1}\chi_4
\end{aligned} \tag{99}$$

$$
\begin{aligned}
[\chi_3, \chi_3] &= \frac{(h+g)^2}{z+1}\chi_1 - 2\frac{h+g}{z+1}\chi_3 + \frac{(h+g)^2}{z+1}\chi_4 \\
[\chi_3, \chi_4] &= \frac{h+g}{z+1}\chi_1 - \frac{1}{z+1}\chi_3 \\
[\chi_4, \chi_1] &= 0 \\
[\chi_4, \chi_2] &= -\frac{1}{z+1}\chi_2 \\
[\chi_4, \chi_3] &= \frac{1}{z+1}\chi_3 - \frac{h+g}{z+1}\chi_4 \\
[\chi_4, \chi_4] &= 0
\end{aligned}
$$

and the commutators are

$$
\begin{aligned}
[\chi_1, \chi_1] &= 0 \\
[\chi_1, \chi_2] &= \chi_1\chi_2 - \left(\frac{z}{z+1}\chi_1\chi_2 + \chi_2\chi_1 + (h+g)\chi_2\chi_2 + \frac{z}{z+1}\chi_4\chi_2\right) \\
[\chi_1, \chi_3] &= \chi_1\chi_3 - \left(\frac{z}{z+1}\chi_1\chi_3 + \frac{z(h+g)}{z+1}\chi_1\chi_4 + \chi_3\chi_1 - (h+g)\chi_3\chi_2 + (h+g)^2\chi_4\chi_2\right. \\
&\qquad \left. - \frac{z}{z+1}\chi_4\chi_3 + \frac{z(h+g)}{z+1}\chi_4\chi_4\right) \\
[\chi_1, \chi_4] &= \chi_1\chi_4 - \chi_4\chi_1 \\
[\chi_2, \chi_1] &= \chi_2\chi_1 - \left(\frac{1}{z+1}\chi_1\chi_2 - (h+g)\chi_2\chi_2 - \frac{z}{z+1}\chi_4\chi_2\right) \\
[\chi_2, \chi_2] &= 0 \\
[\chi_2, \chi_3] &= \chi_2\chi_3 - \left(\frac{z}{z+1}\chi_1\chi_1 + \frac{h+g}{z+1}\chi_1\chi_2 - \frac{z}{z+1}\chi_1\chi_4 - (h+g)^2\chi_2\chi_2 + \chi_3\chi_2\right. \\
&\qquad \left. + \frac{z}{z+1}\chi_4\chi_1 - \frac{(h+g)(2z+1)}{z+1}\chi_4\chi_2 - \frac{z}{z+1}\chi_4\chi_4\right) \\
[\chi_2, \chi_4] &= \chi_2\chi_4 - \left(\frac{z}{z+1}\chi_1\chi_2 + (h+g)\chi_2\chi_2 + \frac{2z+1}{z+1}\chi_4\chi_2\right) \\
[\chi_3, \chi_1] &= \chi_3\chi_1 - \left(-\frac{z(h+g)}{z+1}\chi_1\chi_1 + \frac{2z+1}{z+1}\chi_1\chi_3 - (h+g)^2\chi_2\chi_1 + (h+g)\chi_2\chi_3\right. \\
&\qquad \left. - \frac{z(h+g)}{z+1}\chi_4\chi_1 + \frac{z}{z+1}\chi_4\chi_3\right) \\
[\chi_3, \chi_2] &= \chi_3\chi_2 - \left(-\frac{z}{z+1}\chi_1\chi_1 - \frac{z(h+g)}{z+1}\chi_1\chi_2 + \frac{z}{z+1}\chi_1\chi_4 - (h+g)\chi_2\chi_1\right. \\
&\qquad - (h+g)^2\chi_2\chi_2 + \chi_2\chi_3 + (h+g)\chi_2\chi_4 - \frac{z}{z+1}\chi_4\chi_1 \qquad (100) \\
&\qquad \left. - \frac{z(h+g)}{z+1}\chi_4\chi_2 + \frac{z}{z+1}\chi_4\chi_4\right) \\
[\chi_3, \chi_3] &= \chi_3\chi_3 - \left(\frac{z(h+g)^2}{z+1}\chi_1\chi_1 - \frac{(h+g)(3z+1)}{z+1}\chi_1\chi_3 + \frac{(h+g)^2(2z+1)}{z+1}\chi_1\chi_4\right. \\
&\qquad + (h+g)^3\chi_2\chi_1 - (h+g)^2\chi_2\chi_3 + (h+g)\chi_3\chi_1 - (h+g)^2\chi_3\chi_2 + \chi_3\chi_3 \\
&\qquad - (h+g)\chi_3\chi_4 - \frac{(h+g)^2}{z+1}\chi_4\chi_1 + (h+g)^3\chi_4\chi_2 - \frac{(h+g)(z-1)}{z+1}\chi_4\chi_3
\end{aligned}
$$

$$+\frac{z(h+g)^2}{z+1}\chi_4\chi_4\Big)$$

$$[\chi_3,\chi_4] = \chi_3\chi_4 - \left(\frac{z(h+g)}{z+1}\chi_1\chi_1 - \frac{z}{z+1}\chi_1\chi_3 + (h+g)^2\chi_2\chi_1 - (h+g)\chi_2\chi_3\right.$$
$$\left.+\frac{z(h+g)}{z+1}\chi_4\chi_1 + \frac{1}{z+1}\chi_4\chi_3\right)$$
$$[\chi_4,\chi_1] = \chi_4\chi_1 - \chi_1\chi_4$$
$$[\chi_4,\chi_2] = \chi_4\chi_2 - \left(-\frac{z}{z+1}\chi_1\chi_2 - (h+g)\chi_2\chi_2 + \chi_2\chi_4 - \frac{z}{z+1}\chi_4\chi_2\right)$$
$$[\chi_4,\chi_3] = \chi_4\chi_3 - \left(\frac{z}{z+1}\chi_1\chi_3 - \frac{z(h+g)}{z+1}\chi_1\chi_4 + (h+g)\chi_3\chi_2 + \chi_3\chi_4 - (h+g)^2\chi_4\chi_2\right.$$
$$\left.+\frac{z}{z+1}\chi_4\chi_3 - \frac{z(h+g)}{z+1}\chi_4\chi_4\right)$$
$$[\chi_4,\chi_4] = 0.$$

$\mathcal{L}_2^{4\mathrm{D}}$: The bracket relations are,

$$\begin{aligned}
&[\chi_1,\chi_1] = z\chi_1 - z\chi_4\\
&[\chi_1,\chi_2] = \chi_2\\
&[\chi_1,\chi_3] = z(h+g)\chi_1 - \chi_3 - (z-1)(h+g)\chi_4\\
&[\chi_1,\chi_4] = z\chi_1 - z\chi_4\\
&[\chi_2,\chi_1] = (2z-1)\chi_2\\
&[\chi_2,\chi_2] = 0\\
&[\chi_2,\chi_3] = \chi_1 + (2z-1)(h+g)\chi_2 - \chi_4\\
&[\chi_2,\chi_4] = (2z+1)\chi_2\\
&[\chi_3,\chi_1] = -(z+1)(h+g)\chi_1 + (2z+1)\chi_3 - z(h+g)\chi_4\\
&[\chi_3,\chi_2] = -\chi_1 - (h+g)\chi_2 + \chi_4\\
&[\chi_3,\chi_3] = -(z-1)(h+g)^2\chi_1 + 2(z-1)(h+g)\chi_3 - (z-1)(h+g)^2\chi_4\\
&[\chi_3,\chi_4] = -(z-1)(h+g)\chi_1 + (2z-1)\chi_3 - z(h+g)\chi_4\\
&[\chi_4,\chi_1] = -z\chi_1 + z\chi_4\\
&[\chi_4,\chi_2] = -\chi_2\\
&[\chi_4,\chi_3] = -z(h+g)\chi_1 + \chi_3 + (z-1)(h+g)\chi_4\\
&[\chi_4,\chi_4] = -z\chi_1 + z\chi_4
\end{aligned} \tag{101}$$

and the commutators are,

$$[\chi_1,\chi_1] = \chi_1\chi_1 - (\chi_1\chi_1 + z(h+g)\chi_2\chi_1 - z\chi_2\chi_3 + z\chi_3\chi_2 - z(h+g)\chi_4\chi_2)$$
$$[\chi_1,\chi_2] = \chi_1\chi_2 - (\chi_2\chi_1 + (h+g)\chi_2\chi_2)$$
$$[\chi_1,\chi_3] = \chi_1\chi_3 - (z(h+g)^2\chi_2\chi_1 - z(h+g)\chi_2\chi_3 + \chi_3\chi_1 + (h+g)(z-1)\chi_3\chi_2$$
$$-(h+g)^2(z-1)\chi_4\chi_2)$$
$$[\chi_1,\chi_4] = \chi_1\chi_4 - (z(h+g)\chi_2\chi_1 - z\chi_2\chi_3 + z\chi_3\chi_2 + \chi_4\chi_1 - z(h+g)\chi_4\chi_2)$$
$$[\chi_2,\chi_1] = \chi_2\chi_1 - (-(z-1)\chi_1\chi_2 + z\chi_2\chi_1 + (h+g)(2z-1)\chi_2\chi_2 - z\chi_2\chi_4$$
$$+z\chi_4\chi_2)$$
$$[\chi_2,\chi_2] = 0$$

$$
\begin{aligned}
[\chi_2, \chi_3] &= \chi_2\chi_3 - (-(h+g)(z-1)\chi_1\chi_2 + z(h+g)\chi_2\chi_1 + (h+g)^2(2z-1)\chi_2\chi_2 \\
&\quad -z(h+g)\chi_2\chi_4 + \chi_3\chi_2 + (h+g)(z-1)\chi_4\chi_2) \\
[\chi_2, \chi_4] &= \chi_2\chi_4 - (-z\chi_1\chi_2 + z\chi_2\chi_1 + (h+g)(2z+1)\chi_2\chi_2 - z\chi_2\chi_4 + (z+1)\chi_4\chi_2) \\
[\chi_3, \chi_1] &= \chi_3\chi_1 - ((z+1)\chi_1\chi_3 - z(h+g)\chi_1\chi_4 - (h+g)^2(z+1)\chi_2\chi_1 \\
&\quad +(h+g)(z+1)\chi_2\chi_3 - z\chi_3\chi_1 + z(h+g)\chi_3\chi_2 + z\chi_3\chi_4 + z(h+g)\chi_4\chi_1 \\
&\quad -z(h+g)^2\chi_4\chi_2 - z\chi_4\chi_3) \\
[\chi_3, \chi_2] &= \chi_3\chi_2 - (-(h+g)\chi_2\chi_1 - (h+g)^2\chi_2\chi_2 + \chi_2\chi_3 + (h+g)\chi_2\chi_4) \qquad (102) \\
[\chi_3, \chi_3] &= \chi_3\chi_3 - ((h+g)(z-1)\chi_1\chi_3 - (h+g)^2(z-1)\chi_1\chi_4 - (h+g)^3(z-1)\chi_2\chi_1 \\
&\quad +(h+g)^2(z-1)\chi_2\chi_3 - (h+g)(z-1)\chi_3\chi_1 + (h+g)^2(z-1)\chi_3\chi_2 \\
&\quad +\chi_3\chi_3 + (h+g)(z-1)\chi_3\chi_4 + (h+g)^2(z-1)\chi_4\chi_1 \\
&\quad -(h+g)^3(z-1)\chi_4\chi_2 - (h+g)(z-1)\chi_4\chi_3) \\
[\chi_3, \chi_4] &= \chi_3\chi_4 - (z\chi_1\chi_3 - z(h+g)\chi_1\chi_4 - (h+g)^2(z-1)\chi_2\chi_1 + (h+g)(z-1)\chi_2\chi_3 \\
&\quad -z\chi_3\chi_1 + z(h+g)\chi_3\chi_2 + z\chi_3\chi_4 + z(h+g)\chi_4\chi_1 - z(h+g)^2\chi_4\chi_2 \\
&\quad -(z-1)\chi_4\chi_3) \\
[\chi_4, \chi_1] &= \chi_4\chi_1 - (\chi_1\chi_4 - z(h+g)\chi_2\chi_1 + z\chi_2\chi_3 - z\chi_3\chi_2 + z(h+g)\chi_4\chi_2) \\
[\chi_4, \chi_2] &= \chi_4\chi_2 - (-(h+g)\chi_2\chi_2 + \chi_2\chi_4) \\
[\chi_4, \chi_3] &= \chi_4\chi_3 - (-z(h+g)^2 + z(h+g)\chi_2\chi_3 - (h+g)(z-1)\chi_3\chi_2 + \chi_3\chi_4 \\
&\quad +(h+g)^2(z-1)\chi_4\chi_2) \\
[\chi_4, \chi_4] &= \chi_4\chi_4 - (-z(h+g)\chi_2\chi_1 + z\chi_2\chi_3 - z\chi_3\chi_2 + z(h+g)\chi_4\chi_2 + \chi_4\chi_4).
\end{aligned}
$$

$\mathcal{L}_3^{4\mathrm{D}}$: The bracket relations are,

$$
\begin{aligned}
&[\chi_1, \chi_1] = 0 \\
&[\chi_1, \chi_2] = \chi_2 \\
&[\chi_1, \chi_3] = -\chi_3 + (h+g)\chi_4 \\
&[\chi_1, \chi_4] = 0 \\
&[\chi_2, \chi_1] = -\chi_2 \\
&[\chi_2, \chi_2] = 0 \\
&[\chi_2, \chi_3] = \chi_1 - (h+g)\chi_2 - \chi_4 \\
&[\chi_2, \chi_4] = \chi_2 \qquad (103) \\
&[\chi_3, \chi_1] = -(h+g)\chi_1 + \chi_3 \\
&[\chi_3, \chi_2] = -\chi_1 - (h+g)\chi_2 + \chi_4 \\
&[\chi_3, \chi_3] = (h+g)^2\chi_1 - 2(h+g)\chi_3 + (h+g)^2\chi_4 \\
&[\chi_3, \chi_4] = (h+g)\chi_1 - \chi_3 \\
&[\chi_4, \chi_1] = 0 \\
&[\chi_4, \chi_2] = -\chi_2 \\
&[\chi_4, \chi_3] = \chi_3 - (h+g)\chi_4 \\
&[\chi_4, \chi_4] = 0
\end{aligned}
$$

and the commutators are,

$$
\begin{aligned}
[\chi_1, \chi_1] &= 0 \\
[\chi_1, \chi_2] &= \chi_1\chi_2 - (\chi_2\chi_1 + (h+g)\chi_2\chi_2)
\end{aligned}
$$

$$
\begin{aligned}
[\chi_1, \chi_3] &= \chi_1\chi_3 - (\chi_3\chi_1 - (h+g)\chi_3\chi_2 + (h+g)^2\chi_4\chi_2)\\
[\chi_1, \chi_4] &= 0\\
[\chi_2, \chi_1] &= \chi_2\chi_1 - (\chi_1\chi_2 - (h+g)\chi_2\chi_2)\\
[\chi_2, \chi_2] &= 0\\
[\chi_2, \chi_3] &= \chi_2\chi_3 - ((h+g)\chi_1\chi_2 - (h+g)^2\chi_2\chi_2 + \chi_3\chi_2 - (h+g)\chi_4\chi_2)\\
[\chi_2, \chi_4] &= \chi_2\chi_4 - ((h+g)\chi_2\chi_2 + \chi_4\chi_2)\\
[\chi_3, \chi_1] &= \chi_3\chi_1 - (\chi_1\chi_3 - (h+g)^2\chi_2\chi_1 + (h+g)\chi_2\chi_3)\\
[\chi_3, \chi_2] &= \chi_3\chi_2 - (-(h+g)\chi_2\chi_1 - (h+g)^2\chi_2\chi_2 + \chi_2\chi_3 + (h+g)\chi_2\chi_4) \qquad (104)\\
[\chi_3, \chi_3] &= \chi_3\chi_3 - (-(h+g)\chi_1\chi_3 + (h+g)^2\chi_1\chi_4 + (h+g)^3\chi_2\chi_1 - (h+g)^2\chi_2\chi_3\\
&\quad + (h+g)\chi_3\chi_1 - (h+g)^2\chi_3\chi_2 + \chi_3\chi_3 - (h+g)\chi_3\chi_4 - (h+g)^2\chi_4\chi_1\\
&\quad + (h+g)^3\chi_4\chi_2 + (h+g)\chi_4\chi_3)\\
[\chi_3, \chi_4] &= \chi_3\chi_4 - ((h+g)^2\chi_2\chi_1 - (h+g)\chi_2\chi_3 + \chi_4\chi_3)\\
[\chi_4, \chi_1] &= 0\\
[\chi_4, \chi_2] &= \chi_4\chi_2 - (-(h+g)\chi_2\chi_2 + \chi_2\chi_4)\\
[\chi_4, \chi_3] &= \chi_4\chi_3 - ((h+g)\chi_3\chi_2 + \chi_3\chi_4 - (h+g)^2\chi_4\chi_2)\\
[\chi_4, \chi_4] &= 0.
\end{aligned}
$$

The following result reveals that the relations in the universal enveloping algebras $U(\mathcal{L}_1^{4D})$, $U(\mathcal{L}_2^{4D})$ and $U(\mathcal{L}_3^{4D})$ reflect the structure of the commutation relations of the left-invariant forms presented in theorem 5.5. Indeed the relations in $U(\mathcal{L}_2^{4D})$ and $U(\mathcal{L}_3^{4D})$ are identical and can be obtained from those in $U(\mathcal{L}_1^{4D})$ by setting $z = 0$

Theorem 5.9. The relations in the universal enveloping algebras, $U(\mathcal{L}_1^{4D})$, $U(\mathcal{L}_2^{4D})$ and $U(\mathcal{L}_3^{4D})$ are as follows.

$U(\mathcal{L}_1^{4D})$:

$$
\begin{aligned}
\chi_3\chi_4 &= \frac{z(h+g)}{z+1}\chi_1^2 + (h+g)^2\chi_2\chi_1 - (h+g)\chi_2\chi_3 + \frac{z(h+g)}{z+1}\chi_1\chi_4 - \frac{z}{z+1}\chi_1\chi_3\\
&\quad + \frac{1}{z+1}\chi_4\chi_3 + \frac{h+g}{z+1}\chi_1 - \frac{1}{z+1}\chi_3\\
\chi_3\chi_1 &= -\frac{z(h+g)}{z+1}\chi_1^2 - (h+g)^2\chi_2\chi_1 + (h+g)\chi_2\chi_3 - \frac{z(h+g)}{z+1}\chi_1\chi_4 + \frac{2z+1}{z+1}\chi_1\chi_3\\
&\quad + \frac{z}{z+1}\chi_4\chi_3 - \frac{h+g}{z+1}\chi_1 + \frac{1}{z+1}\chi_3\\
\chi_3\chi_2 &= \frac{z}{z+1}\chi_4^2 - \frac{z}{z+1}\chi_1^2 - (h+g)^2\chi_2^2 - \frac{(2z+1)(h+g)}{z+1}\chi_2\chi_1 \qquad (105)\\
&\quad + \frac{h+g}{z+1}\chi_2\chi_4 + \chi_2\chi_3 - \frac{1}{z+1}\chi_1 - \frac{h+g}{z+1}\chi_2 + \frac{1}{z+1}\chi_4\\
\chi_4\chi_1 &= \chi_1\chi_4\\
\chi_4\chi_2 &= -(h+g)\chi_2^2 - \frac{z}{z+1}\chi_2\chi_1 + \frac{1}{z+1}\chi_2\chi_4 - \frac{1}{z+1}\chi_2\\
\chi_1\chi_2 &= (h+g)\chi_2^2 + \frac{2z+1}{z+1}\chi_2\chi_1 + \frac{z}{z+1}\chi_2\chi_4 + \frac{1}{z+1}\chi_2.
\end{aligned}
$$

$U(\mathcal{L}_2^{4D})$:

$$\begin{aligned}
&\chi_3\chi_4 = (h+g)^2\chi_2\chi_1 - (h+g)\chi_2\chi_3 + \chi_4\chi_3 + (h+g)\chi_1 - \chi_3\\
&\chi_3\chi_1 = -(h+g)^2\chi_2\chi_1 + (h+g)\chi_2\chi_3 + \chi_1\chi_3 - (h+g)\chi_1 + \chi_3\\
&\chi_3\chi_2 = -(h+g)^2\chi_2^2 - (h+g)\chi_2\chi_1 + (h+g)\chi_2\chi_4 + \chi_2\chi_3 - \chi_1 - (h+g)\chi_2 + \chi_4\\
&\chi_4\chi_1 = \chi_1\chi_4\\
&\chi_4\chi_2 = -(h+g)\chi_2^2 + \chi_2\chi_4 - \chi_2\\
&\chi_1\chi_2 = (h+g)\chi_2^2 + \chi_2\chi_1 + \chi_2.
\end{aligned} \tag{106}$$

$U(\mathcal{L}_3^{4D})$:

$$\begin{aligned}
&\chi_3\chi_4 = (h+g)^2\chi_2\chi_1 - (h+g)\chi_2\chi_3 + \chi_4\chi_3 + (h+g)\chi_1 - \chi_3\\
&\chi_3\chi_1 = -(h+g)^2\chi_2\chi_1 + (h+g)\chi_2\chi_3 + \chi_1\chi_3 - (h+g)\chi_1 + \chi_3\\
&\chi_3\chi_2 = -(h+g)^2\chi_2^2 - (h+g)\chi_2\chi_1 + (h+g)\chi_2\chi_4 + \chi_2\chi_3 - \chi_1 - (h+g)\chi_2 + \chi_4\\
&\chi_4\chi_1 = \chi_1\chi_4\\
&\chi_4\chi_2 = -(h+g)\chi_2^2 + \chi_2\chi_4 - \chi_2\\
&\chi_1\chi_2 = (h+g)\chi_2^2 + \chi_2\chi_1 + \chi_2.
\end{aligned} \tag{107}$$

In each case the relations are such that $\{\chi_2^\alpha\chi_1^\beta\chi_4^\gamma\chi_2^\delta : \alpha, \beta, \gamma, \delta \in \mathbb{Z}_{\geqslant 0}\}$ is a basis of the enveloping algebra.

Proof. These relations are obtained by solving the 16 equations $\chi_i\chi_k = \sum_{s,t=1\ldots d}\Lambda_{st,ik}\chi_s\chi_t + \sum_{j=1\ldots d}\mathcal{C}_{ik,j}\chi_j$ for the six quadratic elements $\chi_3\chi_4$, $\chi_3\chi_1$, $\chi_3\chi_2$, $\chi_4\chi_1$, $\chi_4\chi_2$ and $\chi_1\chi_2$. It is then observed that the relations are compatible with the ordering $\chi_2 \prec \chi_1 \prec \chi_4 \prec \chi_3$ so the diamond lemma may be applied to obtain the stated basis. □

6. Three-dimensional bicovariant differential calculi on $SL_h(2)$

Classically (see for example the discussion in the book by Flanders [42]) we obtain the differential calculus on $SL(2)$ from the calculus on $GL(2)$ through the classical relation,

$$\mathrm{d}\mathcal{D} = \mathcal{D}\,\mathrm{Tr}\,\Theta^c \tag{108}$$

where Θ^c is the classical matrix of left-invariant one-forms. $SL(2)$ and its differential calculus is obtained by setting $\mathcal{D} = 1$, so the left-hand side becomes zero and we obtain a linear relation between the classical left-invariant one-forms, namely, $\theta_1 + \theta_4 = 0$. In the standard quantum $GL_q(2)$ case this procedure is *not* possible. The analogue of (108) in this case is rendered trivial by the condition that $\mathcal{D}$ be central in the first-order calculus. More precisely, we have in this case

$$\mathrm{d}\mathcal{D} = \kappa\mathcal{D}\,\mathrm{Tr}_q\,\Theta \tag{109}$$

where $\mathrm{Tr}_q\,\Theta$ is now the q-analogue of $\mathrm{Tr}\,\Theta$, $\theta_1 + q^{-1}\theta_4$. But now, imposing the condition $\mathcal{D} = 1$, immediately fixes $\kappa = 0$ so we have no chance of reducing the dimension of the calculus. There are of course four-dimensional bicovariant calculi on the standard quantum group $SL_q(2)$ but *no* three-dimensional calculi.

In our case, with the nonstandard Jordanian quantum group, the situation is quite different. Studying theorem 5.1 we see that for the calculi Γ_1^{4D} and Γ_3^{4D} we can indeed have the quantum determinant central *and* obtain a dimension-reducing relation through the analogue in these cases of (108),

$$\mathrm{d}\mathcal{D} = \frac{z+2}{2}\mathcal{D}\,\mathrm{Tr}_h\,\Theta. \tag{110}$$

The quantum determinant is central in Γ_1^{4D} and Γ_3^{4D} if and only if the parameter z takes the value 0 or -2. With $z=-2$ we obtain in each case a four-dimensional calculus on $SL_h(2)$, while with $z=0$ the condition $\mathrm{d}\mathcal{D}=0$ yields the linear relation $\mathrm{Tr}_h\,\Theta = \theta_1 + 2h\theta_3 + \theta_4 = 0$—precisely the relation (70) we obtained when we investigated the general implications of choosing a three-dimensional basis of Γ_{inv}. Moreover, $z=0$ is the value of z at which Γ_1^{4D}, Γ_2^{4D} and Γ_3^{4D} coincide, so is already covered as a particular case in Γ_2^{4D}. For this first-order calculus, having set $g=h$, the quantum determinant is central for *all* values of z. In the particular case of $z=\frac{2}{3}$, the differential of the quantum determinant is identically zero so that once again we obtain a four-dimensional calculus on $SL_h(2)$. However, for all other values of z we recover the condition $\mathrm{Tr}_h\,\Theta=0$. At first sight then, it may seem that there is a family of three-dimensional calculi on $SL_h(2)$, but this is not the case.

Theorem 6.1. There is a unique, three-dimensional, first-order bicovariant differential calculus on the Jordanian quantum group $SL_h(2)$, Γ^{3D}. It may be obtained from any one of the three families of first-order bicovariant differential calculi on $GL_{h,g}(2)$ by a reduction analogous to the classical situation. It is specified by its $ABCD$ matrices,

$$A = \begin{pmatrix} 1 & 0 & -h \\ 2h & 1 & h^2 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 2h^2 \\ 0 & 2h & -2h^3 \\ 0 & 0 & -2h \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & h \\ -2h & 1 & -3h^2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{111}$$

Proof. As far as obtaining this calculus from the $GL_{h,g}(2)$ calculi is concerned, we need only observe that starting with the 4×4 $ABCD$ matrices of Γ_3^{4D}, say, in relations (72) and setting $\theta_4 = -\theta_1 - 2h\theta_3$, we obtain commutation relations now involving the 3×3 matrices quoted. That this is the unique three-dimensional calculus on $SL_h(2)$ follows since these are precisely the $ABCD$ matrices we obtain when we apply the procedure of section 4 to look for the most general possible three-dimensional calculus on $SL_h(2)$. □

Now, just as we did for the four-dimensional calculi, we may deduce wedge product commutation relations in the exterior bicovariant graded algebra Ω^{3D}, the Lie brackets and commutators for the quantum Lie algebra $\mathcal{L}^{3D}$, and the enveloping algebra relations for $U(\mathcal{L}^{3D})$. These results are obtained in just the same way as the corresponding results for the four-dimensional calculi so we will not comment on their proofs. We should mention that some results in this direction have already been obtained in [34], where the first-order differential calculus on $SL_h(2)$ was postulated through the R-matrix expression (93). However, the results we present here are more complete than the corresponding results in [34]. Also, it will be useful to collect the results here in our current notation.

Theorem 6.2. The commutation relations between the left-invariant forms in the external bicovariant graded algebra, Ω^{3D}, built upon the first-order calculus Γ^{3D}, are

$$\begin{aligned}
&\theta_3 \wedge \theta_3 = 0 \\
&\theta_3 \wedge \theta_1 = -\theta_1 \wedge \theta_3 \\
&\theta_3 \wedge \theta_2 = -\theta_2 \wedge \theta_3 + 4h\theta_1 \wedge \theta_3 \\
&\theta_1 \wedge \theta_1 = 0 \\
&\theta_1 \wedge \theta_2 = -\theta_2 \wedge \theta_1 - 2h\theta_2 \wedge \theta_3 \\
&\theta_2 \wedge \theta_2 = 4h\theta_2 \wedge \theta_1 + 8h^2\theta_2 \wedge \theta_3.
\end{aligned} \tag{112}$$

The relations are such that $\{\theta_2^\alpha \theta_1^\beta \theta_3^\gamma : \alpha, \beta, \gamma \in \{0, 1\}\}$ is a basis for the exterior algebra of forms.

Theorem 6.3. The quantum Lie brackets and commutators for the quantum Lie algebra $\mathcal{L}^{3D}$, are respectively,

$$\begin{aligned}
&[\chi_1, \chi_1] = 0 \\
&[\chi_1, \chi_2] = 2\chi_2 \\
&[\chi_1, \chi_3] = -2\chi_3 \\
&[\chi_2, \chi_1] = -2\chi_2 \\
&[\chi_2, \chi_2] = 0 \\
&[\chi_2, \chi_3] = \chi_1 - 4h\chi_2 \\
&[\chi_3, \chi_1] = -4h\chi_1 + 2\chi_3 \\
&[\chi_3, \chi_2] = -\chi_1 \\
&[\chi_3, \chi_3] = -4h\chi_3
\end{aligned} \tag{113}$$

and

$$\begin{aligned}
&[\chi_1, \chi_1] = 0 \\
&[\chi_1, \chi_2] = \chi_1\chi_2 - (\chi_2\chi_1 + 4h\chi_2\chi_2) \\
&[\chi_1, \chi_3] = \chi_1\chi_3 - (\chi_3\chi_1 - 4h\chi_3\chi_2) \\
&[\chi_2, \chi_1] = \chi_2\chi_1 - (\chi_1\chi_2 - 4h\chi_2\chi_2) \\
&[\chi_2, \chi_2] = 0 \\
&[\chi_2, \chi_3] = \chi_2\chi_3 - (2h\chi_1\chi_2 - 8h^2\chi_2\chi_2 + \chi_3\chi_2) \\
&[\chi_3, \chi_1] = \chi_3\chi_1 - (\chi_1\chi_3 - 8h^2\chi_2\chi_1 + 4h\chi_2\chi_3) \\
&[\chi_3, \chi_2] = \chi_3\chi_2 - (-2h\chi_2\chi_1 + \chi_2\chi_3) \\
&[\chi_3, \chi_3] = \chi_3\chi_3 - (-2h\chi_1\chi_3 + 2h\chi_3\chi_1 - 8h^2\chi_3\chi_2 + \chi_3\chi_3).
\end{aligned} \tag{114}$$

Theorem 6.4. The relations in the enveloping algebra, $U(\mathcal{L}^{3D})$, are,

$$\begin{aligned}
&\chi_3\chi_1 = -8h^2\chi_2\chi_1 + 4h\chi_2\chi_3 + \chi_1\chi_3 - 4h\chi_1 + 2\chi_3 \\
&\chi_3\chi_2 = -2h\chi_2\chi_1 + \chi_2\chi_3 - \chi_1 \\
&\chi_1\chi_2 = 4h\chi_2^2 + \chi_2\chi_1 + 2\chi_2.
\end{aligned} \tag{115}$$

These relations are such that $\{\chi_2^\alpha \chi_1^\beta \chi_3^\gamma : \alpha, \beta, \gamma \in \mathbb{Z}_{\geqslant 0}\}$ is a basis for $U(\mathcal{L}^{3D})$.

7. The Jordanian quantized universal enveloping algebra

So far we have been working only with the Jordanian quantum analogue of the coordinate ring of $SL_2(\mathbb{C})$. In what follows we focus attention on the corresponding deformation of the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$. Let us recall its definition.

Definition 7.1. The *Jordanian quantized universal enveloping algebra*, $U_h(\mathfrak{sl}_2(\mathbb{C}))$, is the unital associative algebra over $\mathbb{C}[[h]]$ with generators X, Y, H and relations

$$[H, X] = 2\frac{\sinh hX}{h} \tag{116}$$

$$[H, Y] = -Y(\cosh hX) - (\cosh hX)Y \tag{117}$$

$$[X, Y] = H \tag{118}$$

having a basis $\{Y^\alpha H^\beta X^\gamma : \alpha, \beta, \gamma \in \mathbb{Z}_{\geqslant 0}\}$.

In [51] the Casimir element, C, of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ was obtained, in our notation it is

$$C = (Y(\sinh hX) + (\sinh hX)Y) + \tfrac{1}{4}H^2 + \tfrac{1}{4}(\sinh hX)^2. \tag{119}$$

Assuming tensor products to be completed in the h-adic topology, the Hopf structure of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is defined on the generators as,

$$\Delta(X) = X \otimes 1 + 1 \otimes X \tag{120}$$

$$\Delta(Y) = Y \otimes \mathrm{e}^{hX} + \mathrm{e}^{-hX} \otimes Y \tag{121}$$

$$\Delta(H) = H \otimes \mathrm{e}^{hX} + \mathrm{e}^{-hX} \otimes H \tag{122}$$

$$\epsilon(X) = 0 \qquad \epsilon(Y) = 0 \qquad \epsilon(H) = 0 \tag{123}$$

$$S(X) = -X \qquad S(Y) = -\mathrm{e}^{hX}Y\mathrm{e}^{-hX} \qquad S(H) = -\mathrm{e}^{hX}H\mathrm{e}^{-hX}. \tag{124}$$

It is clear that the element $u = \mathrm{e}^{2hX}$ is such that $S^2(x) = uxu^{-1}$ for all $x \in U_h(\mathfrak{sl}_2(\mathbb{C}))$ and $\Delta(u) = u \otimes u$.

$U_h(\mathfrak{sl}_2(\mathbb{C}))$ has received a good deal of attention recently. In particular, we mention the work of Abdesselam *et al* [20] in which a nonlinear map was constructed which realizes the *algebraic* isomorphism between $U_h(\mathfrak{sl}_2(\mathbb{C}))$ and $U(\mathfrak{sl}_2(\mathbb{C}))$. This map was then used by those authors to build the representation theory of $U_h(\mathfrak{sl}_2(\mathbb{C}))$. As with the standard quantization of the enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, the representation theory of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ follows the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ very closely. Indeed, the finite-dimensional, indecomposable representations of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ are in one-to-one correpondence with the finite-dimensional irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$, and can be classified classically by a non-negative half-integer j. Van der Jeugt [21] was able to refine the work of Abdesselam *et al*, obtaining closed-form expressions for the action of the generators of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ on the basis vectors of finite dimensional irreducible representations. Before Van der Jeugt's work Aizawa [52] had demonstrated that the Clebsch–Gordan series for the decomposition of the tensor product of two indecomposable representations of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ was precisely the classical series modulo the one-to-one correspondence of classical and Jordanian representations. Van der Jeugt obtained a general formula for the Clebsch–Gordan coefficients.

8. Jordanian quantum Lie algebra from an ad-submodule in $U_\mathbf{h}(\mathfrak{sl}_2(\mathbb{C}))$

In the following theorem we describe a left ad-submodule of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ which allows us to build a quantum Lie algebra from the enveloping algebra generators.

Theorem 8.1. In $U_h(\mathfrak{sl}_2(\mathbb{C}))$ the space spanned by the elements X_h, H_h and Y_h defined by

$$\begin{aligned}X_h &= \mathrm{e}^{hX}\frac{\sinh hX}{h}\\ H_h &= H\mathrm{e}^{hX}\\ Y_h &= Y\mathrm{e}^{hX} - 2hC\end{aligned} \tag{125}$$

is stable under the left-adjoint action of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ on $U_h(\mathfrak{sl}_2(\mathbb{C}))$.

Proof. To obtain this result, essential use was made of the known PBW basis. With such a basis we can use the computer algebra package REDUCE to perform algebraic manipulations which would be virtually impossible otherwise. In particular, we obtain the following actions of the $U_h(\mathfrak{sl}_2(\mathbb{C}))$ generators on the elements $\{X_h, H_h, Y_h\}$ describing a deformation of the adjoint representation of $U(\mathfrak{sl}_2(\mathbb{C}))$,

$$\begin{aligned}&X \triangleright X_h = 0 \qquad H \triangleright X_h = 2X_h \qquad Y \triangleright X_h = -H_h + 2hX_h\\ &X \triangleright H_h = -2X_h \qquad H \triangleright H_h = 4hX_h \qquad Y \triangleright H_h = 2Y_h + 3h^2X_h\\ &X \triangleright Y_h = H_h \qquad H \triangleright Y_h = -2Y_h - 2hH_h - h^2X_h\\ &Y \triangleright Y_h = -2hY_h - h^2H_h - h^3X_h.\end{aligned} \tag{126}$$

□

The actions of the elements on each other leads to the following *Jordanian quantum Lie brackets* between the elements of the *Jordanian quantum Lie algebra* $\mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C}))$,

$$\begin{aligned}&[X_h, X_h] = 0 \qquad [X_h, H_h] = -2X_h \qquad [X_h, Y_h] = H_h - 2hX_h\\ &[H_h, X_h] = 2X_h, [H_h, H_h] = 0 \qquad [H_h, Y_h] = -2Y_h - 2hH_h + h^2X_h\\ &[Y_h, X_h] = -H_h - 2hX_h \qquad [Y_h, H_h] = 2Y_h - 2hH_h - h^2X_h \qquad [Y_h, Y_h] = -4hY_h\end{aligned} \tag{127}$$

which display the characteristic h-antisymmetry [45]. The $U_h(\mathfrak{sl}_2(\mathbb{C}))$ coproduct on the elements of $\mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C}))$ is,

$$\begin{aligned}\Delta(X_h) &= 1 \otimes X_h + X_h \otimes \mathrm{e}^{2hX}\\ \Delta(H_h) &= 1 \otimes H_h + H_h \otimes \mathrm{e}^{2hX}\\ \Delta(Y_h) &= 1 \otimes Y_h + Y_h \otimes \mathrm{e}^{2hX} + 2h(1 \otimes C + C \otimes \mathrm{e}^{2hX} - \Delta(C)).\end{aligned} \tag{128}$$

A standard definition for the quantum Killing form is the following [46].

Definition 8.2. The *quantum Killing form* is the map $\mathfrak{B} : \mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C})) \otimes \mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C})) \to \mathbb{C}[[h]]$ given by

$$\mathfrak{B}(x, y) = \mathrm{Tr}(xyu) \tag{129}$$

where the trace Tr is taken over the deformed adjoint representation of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ and u is as defined above.

As explained in [46], the Killing form so-defined is ad-invariant, nondegenerate, bilinear and satisfies the following simple generalization of the usual symmetry property,

$$\mathfrak{B}(x, y) = \mathfrak{B}(y, S^2(x)). \tag{130}$$

Note that,

$$\begin{aligned}S^2(X_h) &= X_h\\ S^2(H_h) &= H_h - 4hX_h\\ S^2(Y_h) &= Y_h + 2hH_h - 4h^2X_h\end{aligned} \tag{131}$$

so that $S^2 : \mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C})) \to \mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C}))$.

From the definition it is straightforward to obtain the following evaluations of the quantum Killing form on $\mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C}))$

$$\begin{array}{lll} \mathfrak{B}(H_h, H_h) = 8 & \mathfrak{B}(H_h, X_h) = 0 & \mathfrak{B}(H_h, Y_h) = -8h \\ \mathfrak{B}(X_h, H_h) = 0 & \mathfrak{B}(X_h, X_h) = 0 & \mathfrak{B}(X_h, Y_h) = 4 \\ \mathfrak{B}(Y_h, H_h) = 8h & \mathfrak{B}(Y_h, X_h) = 4 & \mathfrak{B}(Y_h, Y_h) = -6h^2 \end{array} \tag{132}$$

a simple deformation of the classical Killing form, recovered by setting $h = 0$.

9. Jordanian quantum Lie algebra from inverse Clebsch–Gordan coefficients

Classically, for finite-dimensional highest weight modules V_Λ and $V_{\Lambda'}$ of a complex simple Lie algebra $\mathfrak{g}$, there is a decomposition

$$V_\Lambda \otimes V_{\Lambda'} \cong \oplus \mathcal{L}_{\Lambda\Lambda'}^{\Lambda_i} V_{\Lambda_i} \tag{133}$$

of the tensor product module into a direct sum of its irreducible submodules V_{Λ_i}, the non-negative integers $\mathcal{L}_{\Lambda\Lambda'}^{\Lambda_i}$ being the Littlewood–Richardson coefficients of $\mathfrak{g}$. The dimension of the space of intertwiners between $V_\Lambda \otimes V_{\Lambda'}$ and V_{Λ_i} is then just $\mathcal{L}_{\Lambda\Lambda'}^{\Lambda_i}$. In particular, for a pair of adjoint representations, ad, of $\mathfrak{sl}_2(\mathbb{C})$ we have the decomposition

$$\mathrm{ad} \otimes \mathrm{ad} \cong W \oplus \mathrm{ad} \oplus \epsilon \tag{134}$$

where ϵ is the trivial (one-dimensional) representation and W is the irreducible representation of dimension five. Therefore, up to rescaling, there is a unique intertwiner from $\mathrm{ad}\otimes\mathrm{ad} \to \mathrm{ad}$ and a unique intertwiner from $\mathrm{ad} \otimes \mathrm{ad} \to \epsilon$. Indeed, these are precisely the Lie bracket and Killing form of $\mathfrak{sl}_2(\mathbb{C})$, respectively. Choosing bases for the modules W, ad and ϵ, these intertwiners may then be described explicitly by particular subsets of the inverse Clebsch–Gordon coefficients corresponding to the isomorphism (134).

We know that the representations of $U(\mathfrak{sl}_2(\mathbb{C}))$ and $U_h(\mathfrak{sl}_2(\mathbb{C}))$ are in one-to-one correspondence and that the respective tensor structures of these representations are identical modulo this correspondence. Therefore 'quantized' versions of the Lie bracket and Killing form may be obtained from the intertwiners of the corresponding deformed modules. Moreover, by standard theoretical arguments [46] these quantum Lie brackets and Killing form should be identical to those obtained in the last section, up to rescaling. We therefore have a convenient check on the results presented there.

Let us adopt the notation of Van der Jeugt [21]. On the representation space $V_h^{(j)}$ with basis e_m^j where j in a non-negative half-integer and $m = -j, -j+1, \ldots, j$, the action of the generators X, H and Y is given by [21],

$$\begin{aligned} H \triangleright e_m^j &= 2m e_m^j \\ X \triangleright e_m^j &= \sum_{k=0}^{\lfloor (j-m-1)/2 \rfloor} \frac{(h/2)^{2k}}{2k+1} \frac{\alpha_{j,m+1+2k}}{\alpha_{j,m}} e_{m+1+2k}^j \\ Y \triangleright e_m^j &= (j+m)(j-m+1) \frac{\alpha_{j,m-1}}{\alpha_{j,m}} e_{m-1}^j - (j-m)(j+m+1) \left(\frac{h}{2}\right)^2 \frac{\alpha_{j,m+1}}{\alpha_{j,m}} e_{m+1}^j \\ &\quad + \sum_{s=1}^{\lfloor (j-m+1)/2 \rfloor} \left(\frac{h}{2}\right)^{2s} \frac{\alpha_{j,m-1+2s}}{\alpha_{j,m}} e_{m-1+2s}^j \end{aligned} \tag{135}$$

where $\alpha_{j,m} = \sqrt{(j+m)!/(j-m)!}$. Thus the representation matrices of the generators in the deformation of the classical adjoint, $j = 1$, representation are,

$$\Gamma(X) = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad \Gamma(H) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$
$$\Gamma(Y) = \begin{pmatrix} 0 & -\sqrt{2}(h/2)^2 & 0 \\ \sqrt{2} & 0 & -\sqrt{2}(h/2)^2 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{136}$$

The Clebsch–Gordan series for the tensor product of two V_h^1 representations is,

$$V_h^1 \otimes V_h^1 \cong V_h^2 \oplus V_h^1 \oplus V_h^0. \tag{137}$$

If we denote by v_i the ith vector in the ordered basis $\{e_2^2, e_1^2, e_0^2, e_{-1}^2, e_{-2}^2, e_1^1, e_0^1, e_{-1}^1, e_0^0\}$ and by w_i the ith vector in the ordered basis $\{e_1^1 \otimes e_1^1, e_1^1 \otimes e_0^1, e_1^1 \otimes e_{-1}^1, e_0^1 \otimes e_1^1, e_0^1 \otimes e_0^1, e_0^1 \otimes e_{-1}^1, e_{-1}^1 \otimes e_1^1, e_{-1}^1 \otimes e_0^1, e_{-1}^1 \otimes e_{-1}^1\}$, then the Clebsch–Gordan matrix C, where $v_i = \sum_{j=1}^{9} C_{ij} w_j$ is given by [21]

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}h^2}{2\sqrt{3}} & \frac{-h}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{h}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{\sqrt{2}h^2}{2} & -h & \frac{\sqrt{2}h^2}{2} & 0 & \frac{1}{\sqrt{2}} & h & \frac{1}{\sqrt{2}} & 0 \\ \frac{-h^4}{4} & \frac{-\sqrt{2}h^3}{2} & \frac{3h^2}{2} & \frac{\sqrt{2}h^3}{2} & 0 & -\sqrt{2}h & \frac{3h^2}{2} & \sqrt{2}h & 1 \\ -2h & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -h & \frac{1}{\sqrt{2}} & -h & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\sqrt{2}h^2}{2} & -h & \frac{-\sqrt{2}h^2}{2} & 0 & \frac{1}{\sqrt{2}} & -h & \frac{-1}{\sqrt{2}} & 0 \\ \frac{h^2}{\sqrt{3}} & \frac{-\sqrt{2}h}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}h}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \end{pmatrix} \tag{138}$$

with inverse

$$C^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}h & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ \frac{3h^2}{2} & h & \frac{\sqrt{6}}{6} & 0 & 0 & h & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{3}}{3} \\ -\sqrt{2}h & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{-\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{6}}{3} & 0 & 0 & 0 & 0 & 0 & \frac{-\sqrt{3}}{3} \\ \frac{\sqrt{2}h^3}{2} & \frac{\sqrt{2}h^2}{2} & \frac{\sqrt{3}h}{3} & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}h^2}{2} & h & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}h}{3} \\ \frac{3h^2}{2} & -h & \frac{\sqrt{6}}{6} & 0 & 0 & h & \frac{-\sqrt{2}}{2} & 0 & \frac{\sqrt{3}}{3} \\ \frac{-\sqrt{2}h^3}{2} & \frac{\sqrt{2}h^2}{2} & \frac{-\sqrt{3}h}{3} & \frac{\sqrt{2}}{2} & 0 & \frac{-\sqrt{2}h^2}{2} & h & \frac{-\sqrt{2}}{2} & \frac{-\sqrt{6}h}{3} \\ \frac{-h^4}{4} & 0 & \frac{\sqrt{6}h^2}{6} & 0 & 1 & 0 & 0 & 2h & \frac{\sqrt{3}h^2}{3} \end{pmatrix}. \tag{139}$$

Considering $w_i = \sum_{j=1}^{9} C^{-1}_{ij} v_j$ we see that columns 6–8 of C^{-1} correspond to an intertwiner $\mathrm{ad}\otimes\mathrm{ad} \to \mathrm{ad}$ and we deduce a quantum Lie bracket on the vectors $\{e^1_1, e^1_0, e^1_{-1}\}$,

$$\begin{aligned}
&[e^1_1, e^1_1] = 0 \qquad [e^1_1, e^1_0] = \frac{\sqrt{2}}{2} e^1_1 \qquad [e^1_1, e^1_{-1}] = \frac{\sqrt{2}}{2} e^1_0 + h e^1_1 \\
&[e^1_0, e^1_1] = -\frac{\sqrt{2}}{2} e^1_1 \qquad [e^1_0, e^1_0] = 0 \qquad [e^1_0, e^1_{-1}] = \frac{\sqrt{2}}{2} e^1_{-1} + h e^1_0 + \frac{\sqrt{2}}{2} h^2 e^1_1 \\
&[e^1_{-1}, e^1_1] = -\frac{\sqrt{2}}{2} e^1_0 + h e^1_1 \qquad [e^1_{-1}, e^1_0] = -\frac{\sqrt{2}}{2} e^1_{-1} + h e^1_0 - \frac{\sqrt{2}}{2} h^2 e^1_1 \\
&[e^1_{-1}, e^1_{-1}] = 2h e^1_{-1}.
\end{aligned} \tag{140}$$

Similarly, column 9 of C^{-1} corresponds to an intertwiner $\mathrm{ad}\otimes\mathrm{ad} \to \mathbb{C}[[h]]$ and we obtain the Killing form

$$\begin{aligned}
&\mathfrak{B}(e^1_0, e^1_0) = -\frac{\sqrt{3}}{3} \qquad \mathfrak{B}(e^1_0, e^1_1) = 0 \qquad \mathfrak{B}(e^1_0, e^1_{-1}) = \frac{\sqrt{6}h}{3} \\
&\mathfrak{B}(e^1_1, e^1_0) = 0 \qquad \mathfrak{B}(e^1_1, e^1_1) = 0 \qquad \mathfrak{B}(e^1_1, e^1_{-1}) = \frac{\sqrt{3}}{3} \\
&\mathfrak{B}(e^1_{-1}, e^1_0) = -\frac{\sqrt{6}h}{3} \qquad \mathfrak{B}(e^1_{-1}, e^1_1) = \frac{\sqrt{3}}{3} \qquad \mathfrak{B}(e^1_{-1}, e^1_{-1}) = \frac{\sqrt{3}h^2}{3}.
\end{aligned} \tag{141}$$

Now, if we perform the following change of basis,

$$\begin{aligned}
X_h &= 2e^1_1 \\
H_h &= 4he^1_1 - 2\sqrt{2}e^1_0 \\
Y_h &= -\frac{5}{2}h^2 e^1_1 + 2\sqrt{2}h e^1_0 - 2e^1_{-1}
\end{aligned} \tag{142}$$

then the representation matrices of the generators become,

$$\begin{aligned}
&\Gamma(X) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Gamma(H) = \begin{pmatrix} 2 & 4h & -h^2 \\ 0 & 0 & -2h \\ 0 & 0 & -2 \end{pmatrix} \\
&\Gamma(Y) = \begin{pmatrix} 2h & 3h^2 & -h^3 \\ -1 & 0 & -h^2 \\ 0 & 2 & -2h \end{pmatrix}
\end{aligned} \tag{143}$$

which are precisely those obtained from (126) and the Lie bracket relations become (those already obtained in 127),

$$\begin{aligned}
&[X_h, X_h] = 0 \qquad [X_h, H_h] = -2X_h \qquad [X_h, Y_h] = H_h - 2hX_h \\
&[H_h, X_h] = 2X_h, [H_h, H_h] = 0 \qquad [H_h, Y_h] = -2Y_h - 2hH_h + h^2 X_h \\
&[Y_h, X_h] = -H_h - 2hX_h \qquad [Y_h, H_h] = 2Y_h - 2hH_h - h^2 X_h \qquad [Y_h, Y_h] = -4hY_h.
\end{aligned} \tag{144}$$

Further, we can scale the Killing form, by scaling the single basis vector of V^0_h so that on the $\{X_h, H_h, Y_h\}$ it reads

$$\begin{aligned}
&\mathfrak{B}(H_h, H_h) = 8 \qquad \mathfrak{B}(H_h, X_h) = 0 \qquad \mathfrak{B}(H_h, Y_h) = -8h \\
&\mathfrak{B}(X_h, H_h) = 0 \qquad \mathfrak{B}(X_h, X_h) = 0 \qquad \mathfrak{B}(X_h, Y_h) = 4 \\
&\mathfrak{B}(Y_h, H_h) = 8h \qquad \mathfrak{B}(Y_h, X_h) = 4 \qquad \mathfrak{B}(Y_h, Y_h) = -6h^2
\end{aligned} \tag{145}$$

precisely as was found above.

10. Conclusion

Returning to the quantum Lie algebra obtained through Woronowicz's bicovariant calculus, $\mathcal{L}^{3\mathrm{D}}$, if we change the basis according to the identifications,

$$\begin{aligned} H_h &= \chi_1 \\ X_h &= \chi_2 \\ Y_h &= -h\chi_1 + \frac{h^2}{4}\chi_2 + \chi_3 \end{aligned} \tag{146}$$

then the *Woronowicz quantum Lie bracket* on these new basis elements is *precisely* that already found in (127). Thus, as algebras over $\mathbb{C}[[h]]$, the Woronowicz and 'Sudbery–Delius' quantum Lie algebras are *isomorphic*. This means, furthermore, that in addition to having a Killing form we have some natural analogue of the Jacobi identity for this Jordanian quantum lie algebra.

We had already found two appealing aspects of the bicovariant differential geometry on $SL_h(2)$. Namely, its uniqueness and three-dimensionality. The fact that the Woronowicz quantum Lie algebra is isomorphic to the Sudbery–Delius quantum Lie algebra we found starting with $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is a further attractive feature. Recently, the work of Aghamohammadi [53] and Cho *et al* [54] have shown that the corresponding Jordanian quantum plane admits a richer geometrical structure than the standard quantum plane. It should be interesting to try to develop further the geometry on the Jordanian quantum group and also investigate possible $SL(n)$ generalizations.

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